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Worldsheet scattering in $AdS_5 \times S^5$

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Abstract: We calculate the S-matrix in the gauge-fixed sigma-model on $\mathrm{AdS}_5 \times S^5$ to the leading order in perturbation theory, and analyze how supersymmetry is realized on the scattering states. A mild nonlocality of the supercharges implies that their action on multi-particle states does not follow the Leibniz rule, which is replaced by a nontrivial coproduct. The plane wave symmetry algebra is thus naturally enhanced to a Hopf algebra. This structure mirrors that of the large 't Hooft coupling expansion of the S-matrix for the spin chain in the dual super-Yang-Mills theory.

Keywords: AdS-CFT Correspondence, Integrable Field Theories, Quantum Groups.

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Contents

1. Introduction

According to the AdS/CFT duality, type IIB string theory in the $\mathrm{AdS}_5 \times S^5$ background is equivalent to $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions [1]. Understanding and proving the AdS/CFT correspondence requires however solving both the planar limit of $\mathcal{N} = 4$ super-Yang-Mills theory and the $\mathrm{AdS}_5 \times S^5$ worldsheet string theory at finite values of their coupling constants.

While this remains a formidable task, questions of a kinematical nature – such as the determination of their spectra – may be answered by making use of the special properties of these two theories. For $\mathcal{N}=4$ super-Yang-Mills the spectrum of anomalous dimensions of gauge-invariant operators is determined by an auxiliary spin chain whose Hamiltonian is the dilatation operator of the theory. In the appropriate variables, the AdS/CFT correspondence implies that the anomalous dimensions of gauge-invariant operators should equal the worldsheet energies of the corresponding closed string states. Even though both the worldsheet sigma-model [2 − 4] and the spin chain that describes the spectrum of the super-Yang-Mills theory [5, 6] are integrable (see [7] for a review), explicitly solving them is a daunting task.

In the flat space limit the worldsheet theory is free and its spectrum is built from noninteracting oscillators. The curvature and RR flux of $\mathrm{AdS}_5 \times S^5$ introduce nontrivial interactions such that the spectrum is expected to be a complicated collection of discrete levels. However, the integrability of the theory guarantees that the spectrum retains a Fock space structure. Indeed, one of the many definitions of integrability is that one can globally separate action-angle variables and thus define a set of independent oscillators [8].

Although explicitly separating variables is not easy in the quantum theory [9], the features of the outcome of this procedure are quite universal. The spectrum is determined by quantization conditions for a set of particle's momenta which typically constitute a set of coupled functional equations (the Bethe equations [10]). The $2 \to 2$ S-matrix is of central importance for this construction. The S-matrix usually determines the spectrum in an asymptotically large volume and with some additional input the generalization to the exact finite-size spectrum is possible in many cases.

The S-matrix for the super-Yang-Mills spin chain was introduced in [11]. As discussed in [12, 13] the non-perturbative S-matrix is almost completely determined by the global symmetries unbroken by the choice of vacuum state for the spin chain Hamiltonian. An overall abelian phase remains undetermined by symmetries. It was suggested [14] that it should obey a constraint of a similar nature to the crossing symmetry in relativistic quantum field theories.

The first two terms in the large 't Hooft coupling expansion of the abelian phase have been found in [15] and [16], respectively. Subsequently an asymptotic series solution to the crossing condition was constructed in [17]. An analytic continuation to weak coupling, which reproduces the explicit calculation [18] of the four-loop anomalous dimensions of twist-two large spin operators was put forward in a recent paper [19] and further discussed in [20].

The aim of our work is to initiate the perturbative study of the S-matrix of the entire worldsheet sigma-model. Earlier studies, discussing special truncations of the field content of the worldsheet theory, have appeared in [21, 22]. Such calculations have the potential of checking the validity of algebraic considerations for both the tensor structure and the abelian phase of the S-matrix while providing insight into the realization of the symmetries in the interacting theory as well as further confirming its integrability.

Our starting point is the light-cone gauge-fixed worldsheet theory in $\mathrm{AdS}_5 \times S^5$. The Lagrangian has terms with arbitrary numbers of fields of which the quadratic part is that of a free massive theory.[1] The closed string spectrum is the Fock space of massive modes with quantized momenta (BMN modes). The interactions are generated by the geometric curvature and RR-flux; they induce corrections to the free massive spectrum, which have been calculated to leading order in [27] (see also [28, 29]). In the infinite-volume regime the spectrum is continuous and interactions cause a non-trivial scattering of asymptotic states.

We will calculate the worldsheet scattering amplitudes[2] in the light-cone gauge to leading order in perturbation theory. The residual symmetry of the sigma-model in that gauge, the centrally extended $\mathfrak{psu}(2|2) \otimes \mathfrak{psu}(2|2)$, is the same as the symmetry of the spin chain S-matrix [12]. On the worldsheet the central charges arise once the level matching condition is relaxed [30]. As we will show, a mild nonlocality of the supersymmetry generators enhances the symmetry algebra to a Hopf algebra. We will argue that the main consequences of this algebra hold also at the quantum level.

While rigorously proving (quantum) integrability is probably as hard as solving the model exactly, the additional conservation laws present in an integrable theory have testable consequences. In particular, they kinematically forbid particle production in the scattering processes and require factorization of the many-body S-matrix. We will check these properties at tree level for the gauge-fixed sigma-model in $\mathrm{AdS}_5 \times S^5$ by explicit calculations of scattering amplitudes. We should mention that classical integrability (well established for the AdS string) does not automatically guarantee that the corresponding quantum theory is integrable, because conservation laws of higher charges may suffer from quantum anomalies [31]. For the case of the strings in $\mathrm{AdS}_5 \times S^5$ arguments in favor of quantum integrability and the absence of anomalies have been formulated in [4].

We begin in section 2 by describing the field content of the gauge-fixed worldsheet theory and certain puzzling facts about the interplay between its Lagrangian and its expected symmetries. We also summarize our results for the classical S-matrix. In section 3 we derive the action of the symmetry generators on the S-matrix and thus solve the issues raised in the previous section. The two-body S-matrix is calculated to the leading order in perturbation theory in section 4. There we also show that $2 \to 4$ scattering amplitudes vanish for bosonic in- and out-states. In section 5 we calculate the complete tree-level S-matrix, which we compare with the strong-coupling limit of the spin chain S-matrix in section 6. We conclude with the discussion of the results in section 7.

Note added: Arutyunov, Frolov and Zamaklar [55] constructed the S-matrix matrix that satisfies the quantum Yang-Baxter equation and yields in the weak-coupling limit the tree-level scattering matrix found here. As a consequence, the tree-level scattering matrix should obey the classical Yang-Baxter equation. We refer the reader to [55] for the detailed discussion of this important property of the world-sheet S-matrix.

[1] This theory is also the light-cone gauge-fixed string theory in a plane wave which was shown in [23] to be a Penrose limit of $\mathrm{AdS}_5 \times S^5$. It was quantized in [24] and its complete spectrum was constructed in [25]. The relation between the string theory spectrum and gauge-invariant super-Yang-Mills operators was described in detail in [26].

[2] They can only be defined on an infinite string worldsheet and should not be confused with the more familiar target-space amplitudes.

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2. Summary of results

The quantization of the Green-Schwarz string is a longstanding problem and over time various solutions have been proposed, each preserving various parts of the original symmetries of the theory; the more symmetry is preserved the larger the number of unphysical fields appearing in the worldsheet theory. The AdS/CFT correspondence relates gauge theory observables to string theory observables. Consequently, for the purpose of string theory calculations, one is tempted to explicitly eliminate all unphysical degrees of freedom by fixing a unitary gauge. With this motivation in mind we will use the light-cone gauge[3] [29], the fixed-J gauge [27, 34] as well as a one-parameter superposition [35].[4]

The fields. For our purpose it is most convenient to choose the global coordinatization of $\mathrm{AdS}_5 \times S^5$; we will choose the metric

$$ds^2 = -G_{tt}(z)dt^2 + G_{zz}(z)dz^2 + G_{\varphi\varphi}(y)d\varphi^2 + G_{yy}(y)dy^2 \tag{2.1}$$

where

$$G_{tt} = \left(\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}}\right)^2 \quad , \quad G_{zz} = \frac{1}{\left(1-\frac{z^2}{4}\right)^2} \, , \quad G_{\varphi\varphi} = \left(\frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}}\right)^2 \quad , \quad G_{yy} = \frac{1}{\left(1+\frac{y^2}{4}\right)^2} \, . \tag{2.2}$$

y^m and z^μ are four-component vectors. y^2 and z^2 stand for their Euclidean scalar squares. The corresponding worldsheet fields are denoted by capital letters $T, Z, \varPhi, Y$. One combination of the longitudinal fields T and $\varPhi$ will be used in our gauge choice while the derivatives of the other (independent) combination are determined by the Virasoro constraints. As usual in light-cone gauge, its zero-mode is however undetermined.

The $\mathrm{SO}(8) \subset \mathrm{SO}(6) \times \mathrm{SO}(4,2)$ preserved by the gauge choice at the quadratic level is broken by interactions to $\mathrm{SO}(4) \times \mathrm{SO}(4)$. The transverse bosonic fields, Y^m and Z^μ, form the defining representation of this group. A more efficient parametrization in the presence of fermions is provided by the isomorphism $\mathrm{SO}(4) \simeq (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$. Its explicit realization – in terms of the Pauli matrices $\sigma_m = (\mathbb{1}, i\vec{\sigma})$ and $\sigma_\mu = (\mathbb{1}, i\vec{\sigma})$ for the two copies of SO(4) – represents Y and Z as bispinors of the relevant SO(4):

$$Y_{a\dot{a}} = (\sigma_m)_{a\dot{a}} Y^m \quad , \quad Z_{\alpha\dot{\alpha}} = (\sigma_\mu)_{\alpha\dot{\alpha}} Z^\mu \, . \tag{2.3}$$

The fermions also transform as bi-spinors of $\mathrm{SO}(4) \times \mathrm{SO}(4)$, but they are charged with respect to different SU(2) factors. The worldsheet fermions that remain after fixing the κ-symmetry gauge will be denoted by

$$\varPsi_{a\dot{\alpha}} \qquad \text{and} \qquad \varUpsilon_{\alpha\dot{a}} \, . \tag{2.4}$$

[3]There are essentially two ways to fix the light-cone gauge in $\mathrm{AdS}_5 \times S^5$, which differ by picking inequivalent light-cone geodesics. In one case, the light-cone directions lie in AdS_5 [32]; this gauge choice is possible only in the Poincaré patch of AdS_5. In the other case the light cone is shared between AdS_5 and S^5 [33, 29]. We consider the latter case.

[4]Since such gauges (which, incidentally, preserve the least amount of symmetry) typically involve solving the classical constraints of the theory, it is not immediately clear whether any gauge in this class is justified at the quantum level. We are however interested in the classical theory where no subtleties can arise.

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	S^5		AdS_5	
	SU(2)	SU(2)	SU(2)	SU(2)
“Spin” Index	J $a = 1, 2$	$\dot{J}$ $\dot{a} = \dot{1}, \dot{2}$	$\dot{S}$ $\dot{\alpha} = \dot{3}, \dot{4}$	S $\alpha = 3, 4$
$Y_{a\dot{a}}$	**2**	**2**	**1**	**1**
$Z_{\alpha\dot{\alpha}}$	**1**	**1**	**2**	**2**
$\Psi_{a\dot{\alpha}}$	**2**	**1**	**2**	**1**
$\Upsilon_{\alpha\dot{a}}$	**1**	**2**	**1**	**2**

Table 1: $\mathrm{SU}(2)^4$ quantum numbers of the physical degrees of freedom. We use different values for the S^5 part and the AdS_5 part, such that an index can be identified from its value without giving the index symbol. Representations of $\mathrm{SU}(2)^4$ will be denoted by $(\mathbf{2J+1}, \mathbf{2\dot{J}+1}, \mathbf{2\dot{S}+1}, \mathbf{2S+1})$.

The quantum numbers of all fields with respect to $\mathrm{SU}(2)^4$ are summarized in table 1. This description does not fix the action of the supercharges on fields. It turns out [27] that bosons and fermions together form the bi-fundamental representation $((\mathbf{2}|\mathbf{2}), (\mathbf{2}|\mathbf{2}))$ of $\mathrm{PSU}(2|2)_L \times \mathrm{PSU}(2|2)_R$. The bosonic subgroup of each $\mathrm{PSU}(2|2)$ factor consists of two $\mathrm{SU}(2)$ groups, one from each of the original $\mathrm{SO}(4)$ factors. The supercharges relate bosons and fermions following the edges of the diagram:

$$\begin{array}{ccc} Y_{a\dot{a}} & \leftrightarrow & \Psi_{a\dot{\alpha}} \\ \updownarrow & & \updownarrow \\ \Upsilon_{\alpha\dot{a}} & \leftrightarrow & Z_{\alpha\dot{\alpha}} \end{array} \tag{2.5}$$

The odd generators of $\mathrm{PSU}(2|2)_L$ act vertically and the ones of $\mathrm{PSU}(2|2)_R$ act horizontally.

Even though the complete supergroup symmetry is not manifest, one may formally define superindices $A = (a|\alpha)$ and $\dot{A} = (\dot{a}|\dot{\alpha})$, where the lower-case latin indices are Graßmann-even and the greek indices are Graßmann-odd. Thus, all fields combine into a single bi-fundamental supermultiplet of $\mathrm{PSU}(2|2)_L \times \mathrm{PSU}(2|2)_R$ which we will denote by $\Phi_{A\dot{A}}$.

The S-matrix. The two-particle S-matrix is an operator between two copies of the tensor product of the module W_p, generated by $\Phi_{A\dot{A}}(p)$, with itself for different momenta:

$$\$: W_p \otimes W_{p'} \to W_p \otimes W_{p'} \,. \tag{2.6}$$

In the basis provided by $\Phi_{A\dot{A}}(p)$, its matrix representation is

$$\$ \, |\Phi_{A\dot{A}}(p)\Phi_{B\dot{B}}(p')\rangle = |\Phi_{C\dot{C}}(p)\Phi_{D\dot{D}}(p')\rangle \, \$^{C\dot{C}D\dot{D}}_{A\dot{A}B\dot{B}}(p, p') \tag{2.7}$$

and barring anomalies, the S-matrix respects the symmetries of the theory. In an integrable theory the S-matrix satisfies a number of additional kinematic constraints: there should be no particle production and the many-body S-matrix should factorize into the products

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of the two-particle S-matrices. Consistency of the factorization requires that the latter S-matrix satisfies the quantum Yang-Baxter equation (YBE). The YBE is very constraining and in particular a factorizable S-matrix invariant under a non-simple group, such as PSU(2|2) $\times$ PSU(2|2), must be a tensor product of S-matrices for each of the factors (see e.g. [36])[5] :

$$\mathbb{S} = \mathbf{S} \otimes \mathbf{S} \quad , \quad \mathbb{S}_{A\dot{A}B\dot{B}}^{C\dot{C}D\dot{D}}(p,p') = \mathbf{S}_{AB}^{CD}(p,p')\mathbf{S}_{\dot{A}\dot{B}}^{\dot{C}\dot{D}}(p,p') \; . \tag{2.8}$$

It is important to note that a factorized tensor structure does not follow solely from the PSU(2|2) $\times$ PSU(2|2) symmetry considerations. For example, it is in principle possible to scatter a pair of excitations uncharged under the first PSU(2|2) in a singlet combination under the second PSU(2|2) into a pair of excitations uncharged under the second PSU(2|2) in a singlet combination under the first PSU(2|2). In fact, simple inspection of the gauge-fixed Lagrangian yields no hint of the factorized structure (2.8). Confirming group factorization is thus an important test of integrability.

Since only SU(2) $\times$ SU(2) $\subset$ PSU(2|2) is a manifest symmetry of the gauge-fixed world-sheet theory, $\mathbf{S}$ may be parametrized in terms of ten unknown functions of the momenta p and p' of the two incoming particles:[6].

$$\begin{aligned}
\mathbf{S}_{ab}^{cd} &= \mathbf{A}\, \delta_a^c \delta_b^d + \mathbf{B}\, \delta_a^d \delta_b^c \;, & \mathbf{S}_{ab}^{\gamma\delta} &= \mathbf{C}\, \epsilon_{ab}\epsilon^{\gamma\delta} \;, \\
\mathbf{S}_{\alpha\beta}^{\gamma\delta} &= \mathbf{D}\, \delta_\alpha^\gamma \delta_\beta^\delta + \mathbf{E}\, \delta_\alpha^\delta \delta_\beta^\gamma \;, & \mathbf{S}_{\alpha\beta}^{cd} &= \mathbf{F}\, \epsilon_{\alpha\beta}\epsilon^{cd} \;, \\
\mathbf{S}_{a\beta}^{c\delta} &= \mathbf{G}\, \delta_a^c \delta_\beta^\delta \;, & \mathbf{S}_{\alpha b}^{\gamma d} &= \mathbf{L}\, \delta_\alpha^\gamma \delta_b^d \;, \\
\mathbf{S}_{a\beta}^{\gamma d} &= \mathbf{H}\, \delta_a^d \delta_\beta^\gamma \;, & \mathbf{S}_{\alpha b}^{c\delta} &= \mathbf{K}\, \delta_\alpha^\delta \delta_b^c \;.
\end{aligned} \tag{2.9}$$

The first nontrivial order in the expansion of the S-matrix in the sigma-model coupling constant $2\pi/\sqrt{\lambda}$ defines the T-matrix

$$\mathbb{S} = \mathbb{1} + \frac{2\pi i}{\sqrt{\lambda}}\,\mathbb{T} + \mathcal{O}\left(\frac{1}{\lambda}\right) \; . \tag{2.10}$$

The T-matrix should satisfy the classical limit of the YBE (cYBE). Among the restrictions imposed by it is the requirement that the T-matrix inherits the factorized form from the S-matrix:

$$\mathbb{T} = \mathbb{1} \otimes \mathrm{T} + \mathrm{T} \otimes \mathbb{1} \; . \tag{2.11}$$

[5]This can be understood as a requirement that the Faddeev-Zamolodchikov algebra is also a direct product: the field $\Phi_{A\dot{A}}$ is represented by a bilinear in oscillators: $\Phi_{A\dot{A}} \sim z_A z_{\dot{A}}$ each transforming under one of the PSU(2|2) factors. The two sets of oscillators mutually commute. The braiding relations for each of these sets are determined by an PSU(2|2)-invariant S-matrix $\mathbf{S}$ consistent with the Lagrangian of the theory.

[6]These definitions are similar but not identical to those of [12]. The relationship between the two definitions is given in equation (6.15) below.

The components of T are parametrized similar to (2.9) by

$$
\begin{aligned}
\mathrm{T}_{ab}^{cd} &= \mathrm{A}\,\delta_a^c\delta_b^d + \mathrm{B}\,\delta_a^d\delta_b^c\,, & \mathrm{T}_{ab}^{\gamma\delta} &= \mathrm{C}\,\epsilon_{ab}\epsilon^{\gamma\delta}\,,\\
\mathrm{T}_{\alpha\beta}^{\gamma\delta} &= \mathrm{D}\,\delta_\alpha^\gamma\delta_\beta^\delta + \mathrm{E}\,\delta_\alpha^\delta\delta_\beta^\gamma\,, & \mathrm{T}_{\alpha\beta}^{cd} &= \mathrm{F}\,\epsilon_{\alpha\beta}\epsilon^{cd}\,,\\
\mathrm{T}_{a\beta}^{c\delta} &= \mathrm{G}\,\delta_a^c\delta_\beta^\delta\,, & \mathrm{T}_{\alpha b}^{\gamma d} &= \mathrm{L}\,\delta_\alpha^\gamma\delta_b^d\,,\\
\mathrm{T}_{a\beta}^{\gamma d} &= \mathrm{H}\,\delta_a^d\delta_\beta^\gamma\,, & \mathrm{T}_{\alpha b}^{c\delta} &= \mathrm{K}\,\delta_\alpha^\delta\delta_b^c\,.
\end{aligned}
\tag{2.12}
$$

The relation with the coefficients appearing in **S** is given by an expansion similar to (2.10).

A puzzle. Before diffeomorphism and kappa gauge fixing the worldsheet theory is classically integrable; since fixing a unitary gauge may be interpreted as expanding around a classical solution and solving some of the equations of motion, the gauge-fixed theory is expected to be integrable at the classical level. As such, one is entitled to expect that it has a two-particle factorized scattering matrix and that, despite the symmetry algebra being centrally-extended [30], the symmetry transformations act on multi-excitation states via the Leibniz rule.[7] It is moreover usually the case that symmetries fix the tensor structure of the scattering matrix.

Quite surprisingly, the situation at hand is somewhat different: under the assumption of a Leibniz rule action on multi-particle states, the constraints imposed by the symmetry algebra – while qualitatively consistent with the structure of the world sheet Lagrangian – are not consistent with the explicit calculation of the S-matrix.

It appears therefore that the mere existence of a nontrivial (even momentum dependent) center of the symmetry algebra is insufficient to explain the results of worldsheet perturbation theory. The resolution of this puzzle relies on the observation that, even though their action on fields appears at first sight to be local, the $\mathfrak{psu}(2|2)^2$ generators are in fact nonlocal objects. Consequently, their action is subtle and may not follow the Leibniz rule. We will also argue that the nonlocal structure of the symmetry generators is special and it is not affected by perturbative quantum corrections.

The tree-level S-matrix. The T-matrix can be explicitly calculated in perturbation theory. In the gauge where $J_+ = (1-a)J + aE$ is fixed[8] and to leading order in $1/\sqrt{\lambda}$ we

[7] It is worth mentioning that all these expectations are realized in theories with centrally-extended algebras – e.g. WZW models.

[8] Here E is the worldsheet energy and J is the angular momentum on S^5. The constant a is a gauge parameter, which allows one to interpolate between various gauges used in the literature.

found

$$\begin{aligned}
\mathrm{A}(p,p') &= \frac{1}{4}\left[(1-2a)\left(\varepsilon' p-\varepsilon p'\right)+\frac{(p-p')^2}{\varepsilon' p-\varepsilon p'}\right] , \\
\mathrm{B}(p,p') &= -\mathrm{E}(p,p') = \frac{pp'}{\varepsilon' p-\varepsilon p'} , \\
\mathrm{C}(p,p') &= \mathrm{F}(p,p') = \frac{1}{2}\frac{\sqrt{(\varepsilon+1)\,(\varepsilon'+1)}\,(\varepsilon' p-\varepsilon p'+p'-p)}{\varepsilon' p-\varepsilon p'} , \\
\mathrm{D}(p,p') &= \frac{1}{4}\left[(1-2a)\left(\varepsilon' p-\varepsilon p'\right)-\frac{(p-p')^2}{\varepsilon' p-\varepsilon p'}\right] , \\
\mathrm{G}(p,p') &= -\mathrm{L}(p',p) = \frac{1}{4}\left[(1-2a)\left(\varepsilon' p-\varepsilon p'\right)-\frac{p^2-p'^2}{\varepsilon' p-\varepsilon p'}\right] , \\
\mathrm{H}(p,p') &= \mathrm{K}(p,p') = \frac{1}{2}\,\frac{pp'}{\varepsilon' p-\varepsilon p'}\,\frac{(\varepsilon+1)\,(\varepsilon'+1)-pp'}{\sqrt{(\varepsilon+1)\,(\varepsilon'+1)}} .
\end{aligned} \tag{2.13}$$

Here $\varepsilon = \sqrt{1+p^2}$ denotes the relativistic energy. The S-matrix is gauge-dependent, since unlike the spectrum it is not a physical object with clear target-space interpretation. However, the S-matrix determines the spectrum via Bethe equations (at least asymptotically for infinitely long strings) and its gauge-dependence should be simple enough for the solutions of the Bethe equations to be gauge invariant. Indeed, in the class of gauges discussed here, only the diagonal matrix elements are gauge-dependent. The differences between different gauges can be attributed to the gauge dependence of the length of the string [11]. These two effects, the difference in the length and the gauge-dependence of the S-matrix, mutually cancel in the Bethe equations [29, 37].

3. Hopf algebra

The solution to the puzzle described in the previous section and an explanation of the results outlined there turns out to be quite interesting. At its foundation lies the fact that mutual nonlocality of symmetry currents and fundamental fields leads to nontrivial effects which introduce a natural ordering on the fixed-time spacial slices of the worldsheet. This philosophy was applied extensively to the analysis of the nonlocal integrals of motion of relativistic two-dimensional integrable field theories (see e.g. [38, 39]) where it was shown to be equivalent to the YBE. Here we will identify a mild nonlocality of the Noether currents of the $\mathfrak{psu}(2|2)^2$ symmetry and analyze its consequences. We will argue that, in spite of being arrived at through a classical treatment, the technique we use and the basic structure of the result hold unmodified at the quantum level.

Quite generally, given a current J and a field Φ on a $(1+1)$-dimensional worldsheet, their left- and right-multiplications are related by

$$J^A{}_B(x)\,\Phi^C(y) = \Theta^{ACF}_{BDE}\,\Phi^D(y)J^E{}_F(x) \quad \text{if} \quad x>y \ , \tag{3.1}$$

where Θ is usually called the braiding matrix. Obviously, if the current and the field are mutually local the braiding matrix is trivial:

$$\Theta^{ACF}_{BDE} = \delta^A_E\delta^C_D\delta^F_B \ . \tag{3.2}$$

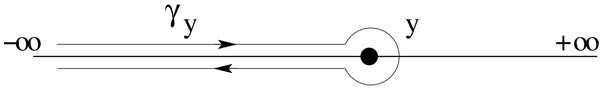

Figure 1: The contour γ_y for the action of the global charges $Q_{(1)}$ on a field inserted at the position y.

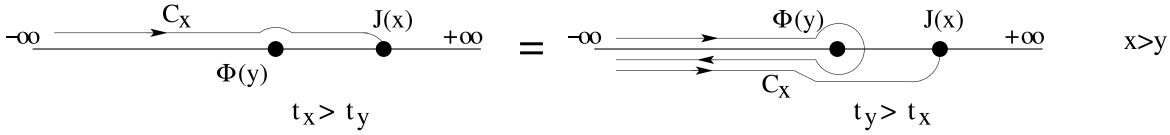

Figure 2: Contour manipulations leading to nontrivial braiding in the product of mutually-nonlocal fields.

However, if J and Φ are mutually nonlocal, then the braiding can be nontrivial. For example, in virtually all theories exhibiting nonlocal conserved charges, the product between the current $J_{(2)}$ whose conserved charge is the first nonlocal charge and the fundamental field of the theory is

$$J^a_{(2)}(x)\,\Phi(y) = \Phi(y)J^a_{(2)}(x) - \tfrac{1}{2} f^{abc}\widehat{Q}^b_{(1)}(\Phi(y))J^c_{(1)}(x) \quad \text{for} \quad x > y\ , \tag{3.3}$$

where a, b and c are adjoint indices, $J_{(1)}$ are the currents for global symmetries, f^{abc} are the structure constants of the corresponding symmetry group and $\widehat{Q}_{(1)}(\Phi(y))$ denotes the usual action of the global symmetries on the fields $\Phi(y)$:

$$\widehat{Q}^b_{(1)}(\Phi(y)) = \int_{\gamma_y} dz J^b_{(1)}(z)\Phi(y) \quad . \tag{3.4}$$

Here the charges acting on fields are defined by integrating the forms dual to the currents along a contour surrounding the point y and not simply along an equal-time slice. The contour γ_y starts and ends at $z = -\infty$ and encircles the point y (cf. figure 1). It is important to note that as we are always considering conserved currents, $\partial_\mu J^a_\mu = 0$, the exact shape of the contour is irrelevant, or in other words, given that we are integrating a closed form the result only depends weakly on the shape of the contour. To understand the origin of the nontrivial braiding matrix Θ in (3.1) let us consider a current J whose definition involves a choice of contour C_x starting at $x = -\infty$ and ending at the location of the current. For any field Φ, the product $J(x)\,\Phi(y)$ comes equipped with the natural time-ordering that a field located to the left of another is also at a later time.[9] Explicitly, $J(x)\,\Phi(y) \equiv J(x, t+\epsilon)\,\Phi(y,t)\big|_{\epsilon\to 0}$ and similarly $\Phi(y)J(x) \equiv \Phi(y, t+\epsilon)J(x,t)\big|_{\epsilon\to 0}$. In this latter case one must make sure that the contour defining $J(x)$ also sits in the past of Φ. Let us then consider the left-hand side of (3.1), $J(x)\,\Phi(y)$ with $x > y$, and rearrange it such that it is in the correct space-like and time-like order. The necessary transformations are illustrated in figure 2. In this figure time runs upward. The left-hand side of figure 2 accounts for the spatial order $J^a_{(2)}(x, t+\epsilon)\,\Phi(y,t) \to \Phi(y,t)J^a_{(2)}(x, t+\epsilon)$. The contour C_x

[9] The reverse choice – that a field located to the left of another is also at an earlier time – may also be made.

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must then be deformed to make sure that, as required by the fact that Φ is located to the left of J, $\Phi(y)$ is always in the future of the contour defining $J(x)$. The appearance of the contour starting and ending at $x = -\infty$ and encircling $\Phi(y)$ is at the origin of the braiding matrix Θ; its precise expression depends on the details of the current J.

In local quantum field theories it is typically the case that the currents corresponding to the global symmetries are local with respect to the fundamental fields of the theory and thus do not exhibit any nontrivial braiding. This is equivalent to the fact that the action of symmetry generators on fields is described by commutators. As we now describe, it turns out that a notable exception to this rule is the worldsheet theory in light-cone gauge, where the nonlocality is provided by the light-cone field x^-.

The gauge-invariant Hamiltonian of the worldsheet sigma-model depends only on the derivatives of x^-; they are determined by the solutions of the constraints and so – order by order in the number of fields – are local operators. As pointed out in [30] (see also appendix B), the $\mathfrak{psu}(2,2|4)$ (super)currents whose supercharges generate $\mathfrak{psu}(2|2)^2$ depend on x^- rather than its derivatives:

$$J_{Q^A{}_B} = e^{i\sigma_{AB}x^-/2}\widetilde{J}_{Q^A{}_B} \qquad \sigma_{AB} = [A] - [B] \qquad x^-(x) = \int_{C_x} dw\, \acute{x}^-(w) \tag{3.5}$$

where $\widetilde{J}$ is a local combination of the transverse fields and $[A]$ denotes the grade of the index A: $[a] = 0$, $[\alpha] = 1$. The contour C_x starts at negative infinity and is the one on the left-hand side of figure 2. Using the fact that the Virasoro constraints imply that

$$\{\acute{x}^-(w),\, \Phi(y)\} = i\,\frac{2\pi}{\sqrt{\lambda}}\delta(w-y)\,\acute{\Phi}(y) \tag{3.6}$$

it is trivial to find, using the same contour manipulations as described in figure 2, that

$$J_{Q^A{}_B}(x)\Phi(y) = \left(e^{-\frac{\pi\sigma_{AB}}{\sqrt{\lambda}}\partial_y}\Phi(y)\right) J_{Q^A{}_B}(x) \quad \text{for} \quad x > y\,. \tag{3.7}$$

In this case the contour deformation is allowed because the integrand of the contour integral is a total derivative.

To find the action of the global symmetry generators on a generic field Φ we use (3.4). Integrating (3.7) the contour γ_z described in figure 1, restoring the indices (3.1) and using the fact that this contour may be split as shown in figure 3 one immediately arrives at the conclusion that the worldsheet supercharges belonging to $\mathfrak{psu}(2|2)^2$ act as follows:

$$\widehat{Q}^A_{(1)\,B}(\Phi^C(y)) = Q^A_{(1)\,B}\Phi^C(y) - \left(e^{-\frac{\pi\sigma_{AB}}{\sqrt{\lambda}}\partial_y}\Phi^C(y)\right) Q^A_{(1)\,B} \tag{3.8}$$

where $Q_{(1)}$ are the usual Noether charges associated to the currents $J_{Q^A{}_B}$.

$$Q^A_{(1)\,B} = \int_{-\infty}^{\infty} dz\, J_{Q^A{}_B}(z) \quad . \tag{3.9}$$

Let us note that, had the $e^{i\sigma_{AB}x^-/2}$ factor been absent from the Noether currents, the equation (3.8) reduced to the usual Poisson bracket action of the Noether charge on fields.

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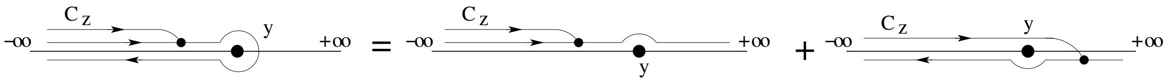

Figure 3: Contour manipulations for the action of a charge on single field.

These arguments can easily be repeated recursively for multi-particle states. For our purpose only two-particle states are of direct interest. Using the same logic as in [38] for the bilocal charges of various integrable field theories, the action of supercharge on a product of fields $\Phi^C(x_1)\Phi^D(x_2)$ requires picking a contour starting and ending at negative infinity and encircling the points x_1 and x_2. The contour is then deformed to separate the action on the two fields; the same arguments as above lead to

$$\widehat{Q}^A_{(1)B}(\Phi^C(x)\Phi^D(x')) = \widehat{Q}^A_{(1)B}(\Phi^C(x))\,\Phi^D(x') + \left(e^{-\frac{\pi\sigma_{AB}}{\sqrt{\lambda}}\partial_x}\Phi^C(x)\right)\widehat{Q}^A_{(1)B}(\Phi^D(x')) \tag{3.10}$$

From a formal algebraic standpoint, this action defines a nontrivial coproduct

$$\Delta(\widehat{Q}^A_{(1)B}) = \widehat{Q}^A_{(1)B}\otimes \mathbb{1} + e^{-\frac{\pi\sigma_{AB}}{\sqrt{\lambda}}\partial_x}\mathbb{1}\otimes\widehat{Q}^A_{(1)B} \tag{3.11}$$

thus promoting the $\mathfrak{psu}(2|2)^2$ to a Hopf algebra (up to a definition of antipode and counit). It is in fact easy to see that this coproduct is precisely that constructed from gauge theory algebraic considerations in [40].[10]

This result represents the resolution of the puzzle described in the previous section. Most importantly, equation (3.11) obviously implies that the $\mathfrak{psu}(2|2)$ generators do not act on products of fields following the Leibniz rule. It is with this coproduct action that the result of the explicit calculation of the T has to be consistent. More precisely, defining the S-matrix as an operator

$$S : W_p\otimes W_{p'} \to W_p\otimes W_{p'} \quad , \tag{3.12}$$

the requirement of invariance under global symmetries translates into [11]

$$\left(\mathbb{1}\otimes\widehat{Q}_{(1)}{}^A{}_B + \widehat{Q}_{(1)}{}^A{}_B\otimes e^{-\frac{i\pi\sigma_{AB}}{\sqrt{\lambda}}p'}\mathbb{1}\right)S = S\left(\widehat{Q}^A_{(1)B}\otimes\mathbb{1} + e^{-\frac{i\pi\sigma_{AB}}{\sqrt{\lambda}}p}\mathbb{1}\otimes\widehat{Q}^A_{(1)B}\right) \tag{3.13}$$

In section 5.4 we will check, to leading order in the $1/\sqrt{\lambda}$ expansion, that this is indeed so. As we will discuss shortly, the conservation of the nonlocal charges should also involve a modified action.

It is worth noting that the details of the coproduct (3.11) depend on the choice of gauge, in particular on the expression of x^- in terms of the transverse fields. This is one source of gauge-dependence of the worldsheet S-matrix and it is the reflection at the algebraic level of the gauge dependence observed in its explicit calculation.

[10] A coproduct implementing the gauge theory symmetry algebra on two-particle spin chain states was constructed in [41]. It is related to the one in [40] by a nonlocal field redefinition.

[11] It is worth noting that, taking expectation values of this equation between two-particle states (located, respectively, $t=+\infty$ and $t=-\infty$), leads to the same constraints on the S-matrix as in the gauge theory analysis.

The arguments used above work just as well at the quantum level provided that no nonlocal contributions to the global symmetry currents are generated by quantum corrections. Assuming that perturbation theory in the light-cone gauge-fixed worldsheet theory is well-defined, it is not hard to construct a two-step argument that this is indeed the case. First, we notice that at any finite order in perturbation theory the relevant part of the Lagrangian is local and it does not depend on x^- but only on its derivatives. Consequently, the (finite or infinite) renormalization of the currents cannot involve x^- and thus must be local (in the sense that they do not require a choice of contour). The second observation is that x^- is the only field exhibiting nontrivial boundary conditions

$$x^-(-\infty) - x^-(+\infty) = p_{\rm ws} \quad . \tag{3.14}$$

From this standpoint it behaves similarly to a soliton whose corresponding topological charge is the worldsheet momentum. Since perturbative effects in a massive theory are local, one may safely expect that they will not affect the action of x^- on local fields.

Putting together these two pieces of argument we reach the conclusion that the structure (3.5) of the Noether supercurrents survives quantum corrections and consequently so does the structure of the coproduct (3.11). Quantum corrections affect only the action of the global charges on single fields $\widehat{Q}^A_{(1)B}(\Phi^C(x_i))$, which is braiding-independent when evaluated on the vacuum.

While formally implying an agreement between gauge and string theory to all orders in perturbation theory (up to gauge artifacts), the discussion above does not directly address the consistency of the resulting S-matrix with integrability. One way to settle this issue is to check whether the S-matrix commutes with the bilocal (and consequently with the higher nonlocal) charge(s).

This is a cumbersome and tedious calculation and we will only briefly outline the necessary steps for defining the action of bilocal charges on the asymptotic states leaving the details of the calculation and the constraints following from them to the interested reader. Evidence for the consistency of the Hopf algebra with integrability in specific examples was previously discussed in e.g. [44]. There it was argued that, while the YBE for the monodromy matrix was not modified, its logarithmic derivative (describing charge conservation) is modified by the inclusion of braiding matrices similar to those in (3.17). The origin of this modification was traced to the rules of differential calculus over the quantum group. It is therefore reasonable to expect that if the asymptotic states form a representation of the Hopf algebra, the YBE is satisfied. It would be interesting to see if it is possible to choose states for the S-matrix described in [12].

As in the case of the conservation of global charges, the conservation of the first nonlocal charge can be expressed as

$$\lim_{T\to\infty} \langle A,p;B,p'|e^{iHT}\Delta(\widehat{Q}_{(2)})|C,p;D,p'\rangle = \lim_{T\to\infty} \langle A,p;B,p'|\Delta(\widehat{Q}_{(2)})e^{iHT}|C,p;D,p'\rangle \tag{3.15}$$

where $\Delta(\widehat{Q}_{(2)})$ denotes the action of $Q_{(2)}$ on a product of two fields. For relativistic field theories with local Noether currents it is known that this analysis leads to the same result as the Yang-Baxter equation [42].

To understand how the action of $Q_{(2)}$ is modified by the presence of the coproduct (3.11) let us first recall that if $J_{(1)}$ denotes the generic global symmetry currents, the first nonlocal charge is

$$
\begin{aligned}
(Q_{(2)})^A{}_B &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{x} dy\, J_{(1)}{}_0^A{}_C(x) J_{(1)}{}_0^C{}_B(y) + \int dx \Sigma^A{}_B(x) \\
&\equiv (Q^{\rm bil}_{(2)})^A{}_B + (Q^{\rm loc}_{(2)})^A{}_B \ ,
\end{aligned}
\tag{3.16}
$$

where Σ is a functional of fields which may be determined by the requirement that $Q_{(2)}$ is conserved. In the absence of kappa-gauge fixing and in conformal gauge Σ has a simple expression in terms of the coset vielbein; classically, in a gauge-fixed framework it is a combination of the space-like components of Noether currents. In a covariant quantization framework it was argued that Σ exists at the quantum level [4].

The expressions of the global symmetry currents depend on the details of both the kappa and diffeomorphism gauges and the expression of Σ inherits this dependence. As for the case of massive relativistic field theories, it is natural to expect that the expression of the bilocal part of $Q_{(2)}$ receives quantum corrections only through the quantum corrections to the expressions of the global symmetry currents while the corrections to Σ however are sensitive to the OPE of the global symmetry currents.

An interesting question is whether it is possible to truncate $Q_{(2)}$ such that it involves only a subalgebra of the full symmetry algebra. Explicit calculation shows that their conservation requires that the currents $J_{(1)}$ represent the complete symmetry algebra; it does not seem possible to truncate J to a subalgebra of $\mathfrak{psu}(2,2|4)$ while maintaining the conservation of $Q_{(2)}$. This appears to complicate the conservation equation (3.15), since some of the generators of $\mathfrak{psu}(2,2|4)$ change the number of excitations of the state they act on. Substantial simplification occurs however if we notice that such effects are irrelevant if we pick two-excitation in- and out-states with both excitations belonging to $\mathfrak{psu}(2|2)^2 \subset \mathfrak{psu}(2,2|4)$. Indeed, states with more than two excitations – potentially created by the action of the components of the currents outside $\mathfrak{psu}(2|2)^2$ – are orthogonal on our chosen out-state. Thus, for the purpose of evaluating the two sides of the equation (3.15) it suffices to consider in (3.16) only the currents $J_{(1)}$ generating $\mathfrak{psu}(2|2)^2$.

The bilocal charge exhibits two kinds on nonlocality and they must be properly taken into account. First, since the currents appearing in $Q^{\rm bil}_{(2)}$ are the global symmetry currents, their x^- dependence introduces a contour C_z similar to that on the left-hand side of figure 2. Secondly, we have the inherent nonlocality of (3.16) which in the absence of the previous contours leads to equation (3.3). The contour associated to the left-most current J_0 in (3.16) ends on this last contour.

Similarly to $Q_{(1)}$, one first finds the action of $Q_{(2)}$ on a single field. The result may then be used to express as a coproduct the action of $Q_{(2)}$ on a product of two fields. The contour manipulations lead to the following structure:

$$
\Delta(\widehat{Q}_{(2)}{}^A{}_B) = \widehat{Q}_{(2)}{}^A{}_B \otimes \mathbb{1} + \Psi_0{}^{AD}_{BC} \otimes \widehat{Q}_{(2)}{}^C{}_D + \Psi_1{}^{AD}_{BC} \otimes \widehat{Q}_{(1)}{}^C{}_D \tag{3.17}
$$

where the formal braiding matrices $\Psi_0{}^{AD}_{BC}$ and $\Psi_1{}^{AD}_{BC}$ include further actions of the global charges as well as of spatial derivatives.

At the classical level, finding the action on states is in principle straightforward. This action however receives quantum corrections and they are currently unknown. Nevertheless, following the example of bosonic sigma-models [43], one may leave them arbitrary and determine them consistently together with the S-matrix. We will however not pursue here this direction and leave it for future work.

The algebraic structure uncovered in the beginning of this section, while of a rather different origin, is similar to that of the gauge theory spin chain. The main difference related to the fact that, even though in [30] the contribution of the zero-mode of x^- to $\mathrm{e}^{\pm ix^-/2}$ was identified as a length-changing operator, the factors $\mathrm{e}^{\pm ix^-/2}$ in the supersymmetry generators act directly on the oscillators building the state rather than by changing the (already infinite) length of the string. In other words, they directly produce momentum-dependent phase factors rather than insert length-changing markers denoted by $\mathcal{Z}^{\pm}$ in [12]. Thus, the phases found on the world sheet are somewhat analogous to those appearing in the nonlocal or cumulative notation of [13]. Indeed, the action of the supercharge on the k-th factor in a product of fields will be multiplied by a phase depending on all momenta of the excitations to the left of this excitation. In the other (twisted, in terminology of [13]) realization of braiding, the $\mathcal{Z}^{\pm}$ markers are crucial for the verification of the YBE as well as for the derivation of the spin chain Bethe equations [12]. It is therefore interesting to see how the Bethe equations arise on the worldsheet.

There is in fact a fairly straightforward procedure to reconstruct the nested Bethe equations given the information already available. To this end it is useful to recall that the usual procedure of constructing the Bethe equations starts with an arbitrary state and imposes that the state is mapped into itself by the scattering of one excitation past all the other ones. Enforcing this condition requires the diagonalization of a product of scattering matrices which is, in fact, the multi-particle S-matrix. For this purpose one chooses an arbitrary type of excitation and treats the states containing only this type of excitation as a new vacuum state; the other excitations are interpreted as excitations above this level-2 vacuum. One then imposes the periodicity condition on these new states. These steps are further repeated until all types of excitations are accounted for. In other words, at each step in the construction of the nested Bethe equations one finds the multi-particle S-matrix with respect to a new vacuum and sets its eigenvalues to unity.

In the presence of the coproduct (3.11), the knowledge of (almost) factorization of a scattering matrix (such that the spin chain S-matrix) following from the YBE allows in principle the construction of all the required multi-particle scattering matrices. The main departure from the usual relation between two-particle and multi-particle S-matrices is the need for additional phases depending on all the excitations building the state. Their appearance can be justified by the fact that, while the 2-particle S-matrix depends only of the two excitations being scattered, the coproduct introduces a phase depending on all the excitation to their left. Thus, these additional phases must be explicitely included in the relation between the multi-particle and two-particle S-matrices.

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4. Scattering of bosons

We start with the bosonic part of the sigma-model action:

$$S_\sigma = \frac{\sqrt{\lambda}}{4\pi} \int d\tau \int_{-\pi}^{\pi} d\sigma \, \sqrt{-h}\, h^{\mathbf{ab}} G_{MN} \partial_{\mathbf{a}} \mathbb{X}^M \partial_{\mathbf{b}} \mathbb{X}^N \,, \tag{4.1}$$

where $\mathbb{X}^M = (T, \varPhi, Y^m, Z^\mu)$. The metric is taken from the equation (2.1). The worldsheet metric $h^{\mathbf{ab}}$ has the signature $(+-)$ and the Levi-Civita symbol $\varepsilon^{\mathbf{ab}}$, used later, is defined such that $\varepsilon^{01} = \varepsilon_{10} = 1$.

4.1 The a-gauge

We consider the gauge in which the angular momentum is uniformly distributed along the string, which is best suited for studying the near-BMN limit [27]. For various purposes it is interesting to look at a one-parameter family of interpolating gauges introduced in [35], which includes the uniform gauge from [27] and its light-cone modification [29] as particular cases. The uniform momentum density in that gauge is associated with

$$J_+ = (1-a)J + aE \tag{4.2}$$

The pure uniform gauge corresponds to $a = 0$, whereas $a = 1/2$ gives the light-cone gauge.

To find the gauge-fixed Lagrangian, we follow the procedure outlined in [45]: T-dualize in the direction canonically conjugate to (4.2), integrate out the worldsheet metric, and fix the gauge in the resulting Nambu-Goto action. The T-duality transformation is facilitated by integrating in a field whose on-shell value is

$$\varPi_{\mathbf{a}}^{(\mathrm{cl})} = \frac{\sqrt{\lambda}}{2\pi} \, \sqrt{-h}\, \varepsilon_{\mathbf{ab}} h^{\mathbf{bc}} \left[(1-a) G_{\varphi\varphi}\, \partial_{\mathbf{c}} \varPhi + a G_{tt}\, \partial_{\mathbf{c}} T \right] \,, \tag{4.3}$$

so that

$$J_+ = \int_{-\pi}^{+\pi} d\sigma \, \varPi_1^{(\mathrm{cl})} \,. \tag{4.4}$$

Adding

$$S_\varPi = \frac{\pi}{\sqrt{\lambda}} \int d^2\sigma \, \frac{\sqrt{-h}\, h^{\mathbf{ab}} \left(\varPi_{\mathbf{a}} - \varPi_{\mathbf{a}}^{(\mathrm{cl})} \right) \left(\varPi_{\mathbf{b}} - \varPi_{\mathbf{b}}^{(\mathrm{cl})} \right)}{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}} \tag{4.5}$$

to the sigma-model action changes nothing since the additional term is quadratic in $\varPi_{\mathbf{a}}$. On the other hand, addition of this term in conjunction with the linear field redefinition

$$T = X^+ - \frac{a}{1-a}\, \varPhi \tag{4.6}$$

eliminates the quadratic dependence on $\varPhi$, leaving only the linear term:

$$S_\varPhi = -\frac{1}{1-a} \int d^2\sigma \, \varepsilon^{\mathbf{ab}} \varPi_{\mathbf{a}} \partial_{\mathbf{b}} \varPhi \,. \tag{4.7}$$

Integrating out $\varPhi$ imposes the constraint $\partial_{\mathbf{a}} \varPi_{\mathbf{b}} - \partial_{\mathbf{b}} \varPi_{\mathbf{a}} = 0$, which is solved by

$$\varPi_{\mathbf{a}} = \partial_{\mathbf{a}} \tilde{\varPhi} \,. \tag{4.8}$$

Substituting this back into the action gives the sigma-model with the T-dual metric [46]:

$$G_{++} = \frac{(1-a)^2 G_{\varphi\varphi} G_{tt}}{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}} \,,$$
$$G_{\tilde{\varphi}\tilde{\varphi}} = \frac{4\pi^2}{\lambda} \, \frac{1}{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}} \,, \tag{4.9}$$

and the B-field

$$B_{\tilde{\varphi}+} = \frac{2\pi}{\sqrt{\lambda}} \, \frac{a G_{tt}}{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}} \,. \tag{4.10}$$

According to (4.8) and (4.4) the dual field satisfies the boundary condition

$$\tilde{\Phi}(\tau, \sigma + 2\pi) = \tilde{\Phi}(\tau, \sigma) + J_+ \,. \tag{4.11}$$

We can now start with the Nambu-Goto action in the T-dual coordinates:

$$S_{NG} = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \, L_{NG}, \tag{4.12}$$

$$L_{NG} = -\sqrt{-\det_{\mathbf{ab}} G_{MN} \, \partial_{\mathbf{a}} \tilde{\mathbb{X}}^M \partial_{\mathbf{b}} \tilde{\mathbb{X}}^N} - \frac{1}{2} \varepsilon^{\mathbf{ab}} B_{MN} \partial_{\mathbf{a}} \tilde{\mathbb{X}}^M \partial_{\mathbf{b}} \tilde{\mathbb{X}}^N, \tag{4.13}$$

where $\tilde{\mathbb{X}}^M = (X^+, \tilde{\Phi}, Y^m, Z^\mu)$. The natural gauge condition, consistent with (4.11), is

$$X^+ = \frac{\tau}{1-a} \quad , \quad \tilde{\Phi} = \frac{J_+ \sigma}{2\pi} \,. \tag{4.14}$$

After imposing this gauge condition it is convenient to rescale σ by $J_+/\sqrt{\lambda}$, so that the worldsheet coordinate changes in the interval $-\pi J_+/\sqrt{\lambda} < \sigma \le \pi J_+/\sqrt{\lambda}$. Then $J_+/\sqrt{\lambda}$ appears only in the length of the string and $\sqrt{\lambda}/2\pi$ enters only as an overall factor in front of the action. We shall further consider the limit $J_+/\sqrt{\lambda} \to \infty$, in which the worldsheet becomes an infinite plane and the dependence on J_+ completely disappears. $2\pi/\sqrt{\lambda}$ remains, as a loop counting parameter.

After all the rescalings, the gauge-fixed Lagrangian does not depend on any parameters at all:

$$\begin{aligned}
L_{\text{g.f.}} = & -\frac{\sqrt{G_{\varphi\varphi} G_{tt}}}{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}} \Bigg\{ 1 - \frac{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}}{2} \\
& \times \left[\left(1 + \frac{1}{G_{\varphi\varphi} G_{tt}}\right) \partial_{\mathbf{a}} X \cdot \partial^{\mathbf{a}} X - \left(1 - \frac{1}{G_{\varphi\varphi} G_{tt}}\right) \left(\dot{X} \cdot \dot{X} + \acute{X} \cdot \acute{X}\right) \right] \\
& + \frac{\left[(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}\right]^2}{2 G_{\varphi\varphi} G_{tt}} \left[(\partial_{\mathbf{a}} X \cdot \partial^{\mathbf{a}} X)^2 - (\partial_{\mathbf{a}} X \cdot \partial_{\mathbf{b}} X)^2 \right] \Bigg\}^{1/2} \\
& + \frac{a}{1-a} \, \frac{G_{tt}}{(1-a)^2 G_{\varphi\varphi} - a^2 G_{tt}} \,.
\end{aligned} \tag{4.15}$$

Here, the index contractions on $X = (Y^m, Z^\mu)$ are done with the metric (2.1). Finally, to the quartic order in the fields we get:

$$\begin{aligned}
L = & \frac{1}{2} (\partial_{\mathbf{a}} X)^2 - \frac{1}{2} X^2 + \frac{1}{4} Z^2 (\partial_{\mathbf{a}} Z)^2 - \frac{1}{4} Y^2 (\partial_{\mathbf{a}} Y)^2 + \frac{1}{4} \left(Y^2 - Z^2\right) \left(\dot{X}^2 + \acute{X}^2\right) \\
& - \frac{1-2a}{8} \left(X^2\right)^2 + \frac{1-2a}{4} (\partial_{\mathbf{a}} X \cdot \partial_{\mathbf{b}} X)^2 - \frac{1-2a}{8} \left[(\partial_{\mathbf{a}} X)^2\right]^2 \,.
\end{aligned} \tag{4.16}$$

Here, unlike in (4.15), target-space indices are contracted with the flat Euclidian metric. The quadratic part of the Lagrangian is 2d Lorentz invariant and SO(8) symmetric. These symmetries are broken by the interaction terms, many of which however preserve SO(8) and/or Lorentz invariance. In particular the gauge-dependent part of the Lagrangian is Lorentz and SO(8) invariant. This part disappears at $a = 1/2$, which reflects relative simplification of the string action in the light-cone gauge [35]. The full action in any a-gauge is only invariant under $\mathfrak{so}(4)^2 = \mathfrak{su}(2)^4$.

4.2 S-matrix

Computing the tree-level S-matrix for the Lagrangian (4.16) is a fairly straightforward exercise. The calculation can be done by applying LSZ reduction to the quartic vertices in (4.16), which produces various tensor structures with the $SO(4)^2$ indices. At the end we want to transform the $SO(4)^2$ vector indices into the $SU(2)^4$ spinor notations according to (2.3), which in effect trades combinations of δ_μ^ν and δ_m^n for combinations of δ_α^β, $\delta_{\dot\alpha}^{\dot\beta}$, δ_a^b and $\delta_{\dot a}^{\dot b}$. The basic SU(2)-invariants are the the identity and the permutation operators:

$$\mathbb{1}_{ab}^{cd} = \delta_b^d \delta_a^c \quad , \quad P_{ab}^{cd} = \delta_b^c \delta_a^d \, , \tag{4.17}$$

and analogous operators acting on the dotted indices. The T-matrix acts in the tensor product and we will use the notations like $\mathbb{1} \otimes P$, $P \otimes \mathbb{1}$ or $P \otimes P$ to denote permutations that act on dotted, undotted or both types of indices. Written in the $SO(4)^2$ notations, these operators parameterize all possible combinations of the SO(4) indices that arise in the scattering amplitudes:

$$\begin{aligned} (\mathbb{1} \otimes P + P \otimes \mathbb{1})_{kl}^{mn} &= \delta_k^m \delta_l^n + \delta_l^m \delta_k^n - \delta^{mn} \delta_{kl} \, , \\ (P \otimes P)_{kl}^{mn} &= \delta_l^m \delta_k^n \, , \\ (\mathbb{1} \otimes \mathbb{1})_{kl}^{mn} &= \delta_k^m \delta_l^n \, . \end{aligned} \tag{4.18}$$

With the use of these formulas we find:

$$\begin{aligned} \mathbb{T}_{YY \to YY} &= \frac{1}{2} \left[(1-2a)(\varepsilon' p - \varepsilon p') + \frac{(p-p')^2}{\varepsilon' p - \varepsilon p'} \right] \mathbb{1} \otimes \mathbb{1} + \frac{pp'}{\varepsilon' p - \varepsilon p'} \left(\mathbb{1} \otimes P + P \otimes \mathbb{1} \right) \, , \\ \mathbb{T}_{ZZ \to ZZ} &= \frac{1}{2} \left[(1-2a)(\varepsilon' p - \varepsilon p') - \frac{(p-p')^2}{\varepsilon' p - \varepsilon p'} \right] \mathbb{1} \otimes \mathbb{1} - \frac{pp'}{\varepsilon' p - \varepsilon p'} \left(\mathbb{1} \otimes P + P \otimes \mathbb{1} \right) \, , \\ \mathbb{T}_{ZY \to ZY} &= \frac{1}{2} \left[(1-2a)(\varepsilon' p - \varepsilon p') + \frac{p^2 - p'^2}{\varepsilon' p - \varepsilon p'} \right] \, , \\ \mathbb{T}_{ZY \to YZ} &= 0 \, , \\ \mathbb{T}_{ZZ \to YY} &= 0 \, . \end{aligned} \tag{4.19}$$

The (bosonic) T-matrix appears to have a factorized form (2.11). We should emphasize that this is a result of very delicate cancelations among different diagrams. From (4.19) we can extract coefficients A, B, D, E, G and L in (2.12), see (2.13). C, F, H, and K only appear in the scattering of fermions.

4.3 Absence of particle production

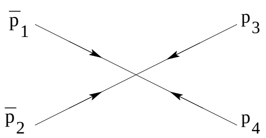
In this section we offer some arguments for the factorization of the bosonic S-matrix beyond leading order, in particular the absence of $2 \to 4$ particle production at tree level and the corresponding factorization of the $3 \to 3$ tree level amplitude. It is well known that the factorization of the S-matrix follows from the selection rules that the number of particles of a given mass is unchanged and that the final momenta are the same as the initial ones, see for example [47]. It is straightforward to keep higher terms in the expansion of the light-cone Lagrangian provided we restrict our attention to the bosonic part. Using the uniform light-cone gauge $a = \frac{1}{2}$ for convenience we find the Lagrangian density describing only fields on the S^5

$$\begin{aligned}\mathcal{L}_{\rm lc} &= P_y \dot{Y} - \mathcal{H}_{\rm lc} \\ &= -\frac{1}{2}\left(-\dot{Y}^2 + \acute{Y}^2 + Y^2\right) + \frac{1}{2\sqrt{\lambda}} Y^2 \acute{Y}^2 \\ &\quad + \frac{1}{32\lambda}\left(-Y^2\acute{Y}^4 + Y^4\dot{Y}^2 - Y^2\dot{Y}^4 - \acute{Y}^2(9Y^4 + 2Y^2\dot{Y}^2) + 4Y^2(\dot{Y}\cdot\acute{Y})^2\right) + \dots\end{aligned} \tag{4.20}$$

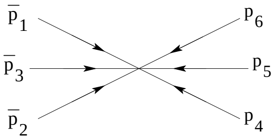
and the analogous Lagrangian density for fields on the AdS_5 is

$$\begin{aligned}\mathcal{L}_{\rm lc} &= -\frac{1}{2}\left(-\dot{Z}^2 + \acute{Z}^2 + Z^2\right) - \frac{1}{2\sqrt{\lambda}} Z^2 \acute{Z}^2 \\ &\quad + \frac{1}{32\lambda}\left(-Z^2\acute{Z}^4 + Z^4\dot{Z}^2 - Z^2\dot{Z}^4 - \acute{Z}^2(9Z^4 + 2Z^2\dot{Z}^2) + 4Z^2(\dot{Z}\cdot\acute{Z})^2\right) + \dots\end{aligned} \tag{4.21}$$

The dots refer to terms of higher order in $1/\sqrt{\lambda}$. The mixed terms can also be easily found but we do not record them here. We further restrict our attention to two directions on the sphere, Y^5, Y^6, and consider the scattering of the complex coordinate $Y = \frac{1}{2}(Y^5 + iY^6)$. The vertices for the above interactions are given by



$$= \frac{-i}{\sqrt{\lambda}}(p_1 + p_2)(p_3 + p_4) \tag{4.22}$$



$$\begin{aligned}= \frac{i}{32\lambda}\Big[&8(\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3 - (p_1p_2 + p_1p_3 + p_2p_3))\times \\ &(\varepsilon_4\varepsilon_5 + \varepsilon_4\varepsilon_6 + \varepsilon_5\varepsilon_6 - (p_4p_5 + p_4p_6 + p_5p_6)) \\ &+8(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)(\varepsilon_4 + \varepsilon_5 + \varepsilon_6) - 64(p_1 + p_2 + p_3)(p_4 + p_5 + p_6)\Big]\end{aligned} \tag{4.23}$$

where the two-momenta are the pairs (ε_i, p_i). The contributions to the $2 \to 4$ scattering involving two four-vertices are given by

$$\mathbf{A}(4,5,6): \quad \frac{1}{\sqrt{\lambda}}\left(\frac{(p_1+p_2)^2(p_5+p_6)^2}{(\varepsilon_1+\varepsilon_2-\varepsilon_4)^2 - (p_1+p_2-p_4)^2 - 1}\right) + (4 \leftrightarrow 5) + (4 \leftrightarrow 6) \tag{4.24}$$

$$\mathbf{B}(4,5,6): \quad \frac{1}{\sqrt{\lambda}}\left(\frac{(p_1-p_3)^2(p_5+p_6)^2}{(\varepsilon_1-\varepsilon_3-\varepsilon_4)^2 - (p_1-p_3-p_4)^2 - 1}\right) + (4 \leftrightarrow 5) + (4 \leftrightarrow 6) \tag{4.25}$$

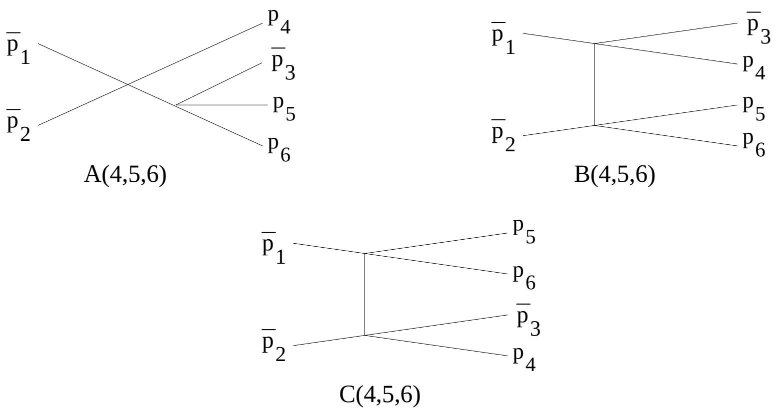


Figure 4: Three of the diagrams contributing to $2 \to 4$ scattering.

$$\mathbf{C}(4,5,6): \quad \frac{1}{\sqrt{\lambda}}\left(\frac{(p_2-p_3)^2(p_5+p_6)^2}{(\varepsilon_1-\varepsilon_5-\varepsilon_6)^2-(p_1-p_5-p_6)^2-1}\right)+(4\leftrightarrow 5)+(4\leftrightarrow 6) \quad (4.26)$$

and from the six-vertex we get the contribution

$$\begin{aligned}\mathbf{D}: \quad \frac{i}{32\lambda}&\,(8(\varepsilon_1\varepsilon_2-\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_3-(p_1p_2-p_1p_3-p_2p_3)) \times \\ &(\varepsilon_4\varepsilon_5+\varepsilon_4\varepsilon_6+\varepsilon_5\varepsilon_6-(p_4p_5+p_4p_6+p_5p_6)) \\ &-8(\varepsilon_1+\varepsilon_2-\varepsilon_3)(\varepsilon_4+\varepsilon_5+\varepsilon_6)+64(p_1+p_2-p_3)(p_4+p_5+p_6)) \ . \qquad (4.27)\end{aligned}$$

We can now (analytically and numerically) check that $A(4,5,6)+B(4,5,6)+C(4,5,6)+(4\leftrightarrow 5)+(4\leftrightarrow 6)+D=0$ for generic values of the external momenta. We can see this explicitly in some simple cases; for example set $p_1=-p_2, p_5=-p_6$. In this case $p_3=-p_4$ and $\varepsilon_1=\varepsilon_3+\varepsilon_5$ (on-shell $\varepsilon_i=\sqrt{1+p_i^2}$) and we can see that all diagrams of type A vanish as do $B(4,5,6)$ and $C(4,5,6)$. The remaining contribution from $B(5,6,4)$, $B(6,5,4)$, $C(5,6,4)$ and $C(6,5,4)$ can be simplified using

$$p_5=\left(\left(\sqrt{1+p_1^2}-\sqrt{1+p_3^2}\right)^2-1\right)^{\frac{1}{2}} \qquad (4.28)$$

to

$$-\frac{i}{\sqrt{\lambda}}\left(p_1^4+(p_1^2+3p_3^2)+p_1^2(p_3^3-2\varepsilon_1\varepsilon_3)\right) \ . \qquad (4.29)$$

Using energy and momentum conservation it is straightforward to show that the six-vertex gives the negative of this result. We can also examine the case when all the momenta are much larger than the mass and again $p_1=-p_2$ in this case $\varepsilon_i \simeq |p_i|$ and by examining specific cases it is straightforward to see that the $2 \to 4$ amplitude vanishes. Thus we have shown complete cancellation between the diagrams for $2 \to 4$ scattering.

In fact we have shown more than the absence of particle production, if we consider $3 \to 3$ scattering we find the exact same cancellations as above except for the special kinematical region where the outgoing momenta are equal to the incoming. In this case the internal propagator in the two vertex diagrams become singular giving an amplitude which splits into a finite part canceled by the six-vertex term and momentum δ-function. Hence we also see the factorization of the $3 \to 3$ tree level amplitude into $2 \to 2$ events which in this case is equivalent to the absence of particle production.

We should mention that the authors of [29] were able to construct a unitary transformation which, quite generically, removed particle producing terms from the light-cone Hamiltonian. The existence of such a transformation does not require any particular symmetries of the Hamiltonian but only relies upon the existence of the quantum theory and the non-zero mass of the free particles. Indeed the remaining terms in the Hamiltonian can have quite generic coefficients and so in this case the absence of particle production does not seem to necessarily imply the factorization of multi-particle scattering processes. This is an important distinction as it is this factorization which is equivalent to the existence of higher conserved charges and so integrability.

5. Scattering of fermions

5.1 Physical degrees of freedom

We now turn to the scattering of fermions. For the sake of simplicity we shall only consider the uniform light-cone gauge that corresponds to $a = 1/2$ in (4.2). The results for the constant-J gauge ($a = 0$) are displayed in appendix B. The degrees of freedom that are left after fixing the uniform light-cone gauge are given by the fields $Y_{a\dot{a}}$, $Z_{\alpha\dot{\alpha}}$, $\Psi_{a\dot{\alpha}}$ and $\Upsilon_{\alpha\dot{a}}$. See section 2 and table 1 for more details.

We use northeast-southwest conventions to raise and lower $\mathfrak{su}(2)$ indices

$$x^a = \epsilon^{ab} x_b \quad , \quad x_a = x^b \epsilon_{ba} \; , \tag{5.1}$$

where $\epsilon^{12} = \epsilon_{12} = 1$, and likewise for all other indices. Also complex conjugation changes the position of the index, e.g. $(Y_{a\dot{a}})^* \equiv Y^{*a\dot{a}}$. It is important not to confuse this with $Y^*_{a\dot{a}} \equiv Y^{*b\dot{b}} \epsilon_{ba} \epsilon_{\dot{b}\dot{a}}$. Moreover, the bosonic fields satisfy the reality condition

$$Y^*_{a\dot{a}} = Y_{a\dot{a}} \qquad \text{and} \qquad Z^*_{\alpha\dot{\alpha}} = Z_{\alpha\dot{\alpha}} \; . \tag{5.2}$$

5.2 Action and quantization

For the superstring calculation in uniform light-cone gauge, we use the action derived in [29]. In appendix C we rewrite this action in a second order formalism and obtain

$$\mathcal{S} = \sqrt{\lambda} \int_{-\infty}^{\infty} d\tau \int_{-\pi J_+/\sqrt{\lambda}}^{\pi J_+/\sqrt{\lambda}} \frac{d\sigma}{2\pi} \, \mathcal{L} \tag{5.3}$$

with

$$\mathcal{L}_0 = \operatorname{str}\left[\frac{1}{4}\dot{X}\dot{X} - \frac{1}{4}\acute{X}\acute{X} - \frac{1}{4}XX - \frac{i}{2}\varSigma_+\chi\dot{\chi} - \frac{1}{2}\varSigma_+\chi\acute{\chi}^\natural - \frac{1}{2}\chi\chi\right] ,$$

$$\begin{aligned}
\mathcal{L}_{\text{int}} = & -\frac{1}{8}\operatorname{str}\varSigma_8 XX \operatorname{str}\acute{X}\acute{X} \\
& + \frac{1}{8}\operatorname{str}\chi\acute{\chi}\chi\acute{\chi} + \frac{1}{8}\operatorname{str}\chi\chi\acute{\chi}\acute{\chi} + \frac{1}{16}\operatorname{str}[\chi,\acute{\chi}][\chi^\natural,\acute{\chi}^\natural] + \frac{1}{4}\operatorname{str}\chi\acute{\chi}^\natural\chi\acute{\chi}^\natural \\
& - \frac{1}{8}\operatorname{str}\varSigma_8 XX \operatorname{str}\acute{\chi}\acute{\chi} + \frac{1}{4}\operatorname{str}[X,\acute{X}][\chi,\acute{\chi}] + \operatorname{str}X\acute{\chi}X\acute{\chi} \\
& + \frac{i}{8}\operatorname{str}[X,\dot{X}][\chi^\natural,\acute{\chi}] - \frac{i}{8}\operatorname{str}[X,\dot{X}][\chi,\acute{\chi}^\natural] .
\end{aligned} \tag{5.4}$$

The conjugation $\chi^\natural$ as well as the constant matrices $\varSigma_+$ and $\varSigma_8$ are defined in the appendix C. This action contains only the physical fields introduced above. They are written as elements of two $\mathrm{SU}(2,2|4)$ supermatrices. The bosons are contained in

$$X = \left(\begin{array}{cccc|cccc}
0 & 0 & +Z^{3\dot{4}} & +iZ^{3\dot{3}} & 0 & 0 & 0 & 0 \\
0 & 0 & +iZ^{4\dot{4}} & -Z^{4\dot{3}} & 0 & 0 & 0 & 0 \\
-Z^{4\dot{3}} & -iZ^{3\dot{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\
-iZ^{4\dot{4}} & +Z^{3\dot{4}} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & +iY^{1\dot{2}} & -Y^{1\dot{1}} \\
0 & 0 & 0 & 0 & 0 & 0 & -Y^{2\dot{2}} & -iY^{2\dot{1}} \\
0 & 0 & 0 & 0 & -iY^{2\dot{1}} & +Y^{1\dot{1}} & 0 & 0 \\
0 & 0 & 0 & 0 & +Y^{2\dot{2}} & +iY^{1\dot{2}} & 0 & 0
\end{array}\right) , \tag{5.5}$$

and the fermions in

$$\chi = e^{\frac{i\pi}{4}} \left(\begin{array}{cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & +\varUpsilon^{3\dot{2}} & +i\varUpsilon^{3\dot{1}} \\
0 & 0 & 0 & 0 & 0 & 0 & +i\varUpsilon^{4\dot{2}} & -\varUpsilon^{4\dot{1}} \\
0 & 0 & 0 & 0 & +i\varPsi^{*2\dot{3}} & -\varPsi^{*1\dot{3}} & 0 & 0 \\
0 & 0 & 0 & 0 & -\varPsi^{*2\dot{4}} & -i\varPsi^{*1\dot{4}} & 0 & 0 \\
\hline
0 & 0 & +\varPsi^{1\dot{4}} & +i\varPsi^{1\dot{3}} & 0 & 0 & 0 & 0 \\
0 & 0 & +i\varPsi^{2\dot{4}} & -\varPsi^{2\dot{3}} & 0 & 0 & 0 & 0 \\
-i\varUpsilon^{*4\dot{1}} & +\varUpsilon^{*3\dot{1}} & 0 & 0 & 0 & 0 & 0 & 0 \\
+\varUpsilon^{*4\dot{2}} & +i\varUpsilon^{*3\dot{2}} & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) . \tag{5.6}$$

Plugging in these expression, the free part of the Lagrangian becomes

$$\begin{aligned}
\mathcal{L}_0 = & + \tfrac{1}{2}\dot{Y}^*_{a\dot{a}}\dot{Y}^{a\dot{a}} - \tfrac{1}{2}\acute{Y}^*_{a\dot{a}}\acute{Y}^{a\dot{a}} - \tfrac{1}{2}Y^*_{a\dot{a}}Y^{a\dot{a}} \\
& + \tfrac{1}{2}\dot{Z}^*_{\alpha\dot{\alpha}}\dot{Z}^{\alpha\dot{\alpha}} - \tfrac{1}{2}\acute{Z}^*_{\alpha\dot{\alpha}}\acute{Z}^{\alpha\dot{\alpha}} - \tfrac{1}{2}Z^*_{\alpha\dot{\alpha}}Z^{\alpha\dot{\alpha}} \\
& + i\varPsi^*_{a\dot{\alpha}}\dot{\varPsi}^{a\dot{\alpha}} - \tfrac{i}{2}(\varPsi^*_{a\dot{\alpha}}\acute{\varPsi}^{*a\dot{\alpha}} + \varPsi_{a\dot{\alpha}}\acute{\varPsi}^{a\dot{\alpha}}) - \varPsi^*_{a\dot{\alpha}}\varPsi^{a\dot{\alpha}} \\
& + i\varUpsilon^*_{\alpha\dot{a}}\dot{\varUpsilon}^{\alpha\dot{a}} - \tfrac{i}{2}(\varUpsilon^*_{\alpha\dot{a}}\acute{\varUpsilon}^{*\alpha\dot{a}} + \varUpsilon_{\alpha\dot{a}}\acute{\varUpsilon}^{\alpha\dot{a}}) - \varUpsilon^*_{\alpha\dot{a}}\varUpsilon^{\alpha\dot{a}} .
\end{aligned} \tag{5.7}$$

The corresponding equations of motion are solved by the following mode expansion:

$$Y_{a\dot{a}}(\vec{x}) = \int \frac{dp}{2\pi} \frac{1}{\sqrt{2\varepsilon}} \left(a_{a\dot{a}}(p)\, e^{-i\vec{p}\cdot\vec{x}} + a^\dagger_{a\dot{a}}(p)\, e^{+i\vec{p}\cdot\vec{x}} \right) , \tag{5.8}$$

$$Z_{\alpha\dot{\alpha}}(\vec{x}) = \int \frac{dp}{2\pi} \frac{1}{\sqrt{2\varepsilon}} \left(a_{\alpha\dot{\alpha}}(p)\, e^{-i\vec{p}\cdot\vec{x}} + a^\dagger_{\alpha\dot{\alpha}}(p)\, e^{+i\vec{p}\cdot\vec{x}} \right) , \tag{5.9}$$

$$\Psi_{a\dot{\alpha}}(\vec{x}) = \int \frac{dp}{2\pi} \frac{1}{\sqrt{\varepsilon}} \left(b_{a\dot{\alpha}}(p)\, u(p)\, e^{-i\vec{p}\cdot\vec{x}} + b^\dagger_{a\dot{\alpha}}(p)\, v(p)\, e^{+i\vec{p}\cdot\vec{x}} \right) , \tag{5.10}$$

$$\Upsilon_{\alpha\dot{a}}(\vec{x}) = \int \frac{dp}{2\pi} \frac{1}{\sqrt{\varepsilon}} \left(b_{\alpha\dot{a}}(p)\, u(p)\, e^{-i\vec{p}\cdot\vec{x}} + b^\dagger_{\alpha\dot{a}}(p)\, v(p)\, e^{+i\vec{p}\cdot\vec{x}} \right) , \tag{5.11}$$

where the energy is $\varepsilon = \sqrt{1+p^2}$, the wave functions are

$$u(p) = \cosh\tfrac{\theta}{2} \quad , \quad v(p) = \sinh\tfrac{\theta}{2} \tag{5.12}$$

and the rapidity θ is defined through $p = \sinh\theta$. The scalar product in the exponentials is $\vec{p}\cdot\vec{x} = \varepsilon\tau + p\sigma$. The canonical commutation relations are given by

$$\begin{aligned} &[a^{a\dot{a}}(p), a^\dagger_{b\dot{b}}(p')] = 2\pi\, \delta^a_b \delta^{\dot{a}}_{\dot{b}}\, \delta(p-p') \,, && \{b^{a\dot{\alpha}}(p), b^\dagger_{b\dot{\beta}}(p')\} = 2\pi\, \delta^a_b \delta^{\dot{\alpha}}_{\dot{\beta}}\, \delta(p-p') \,, \\ &[a^{\alpha\dot{\alpha}}(p), a^\dagger_{\beta\dot{\beta}}(p')] = 2\pi\, \delta^\alpha_\beta \delta^{\dot{\alpha}}_{\dot{\beta}}\, \delta(p-p') \,, && \{b^{\alpha\dot{a}}(p), b^\dagger_{\beta\dot{b}}(p')\} = 2\pi\, \delta^\alpha_\beta \delta^{\dot{a}}_{\dot{b}}\, \delta(p-p') \,. \end{aligned} \tag{5.13}$$

The above choice of labeling the modes has some very nice features. Firstly, bosons and fermions are treated identically. All indices are carried by the mode operators. The wave functions are scalar functions and no Dirac matrices are required. Secondly, particles and anti-particles can be considered at once without notational differences. The particle/anti-particle relationship is determined by which pair of oscillators occurs in the expansion of one field. Looking, for example, at the field $Y_{a\dot{a}}$ (for fixed a and $\dot{a}$), we see that the oscillator $a^\dagger_{a\dot{a}}$ creates the "anti-excitation" of the "excitation" that is destroyed by the oscillator $a_{a\dot{a}}$. Note, however, that these two oscillators do not form a canonical pair. Rather $a^\dagger_{a\dot{a}}$ and $a^{a\dot{a}} = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}} a_{b\dot{b}}$ are conjugate to each other as it can be seen from the commutation relations. This is after all a consequence of $(a^{a\dot{a}})^* = a^\dagger_{a\dot{a}}$. It is interesting to observe the different origin of the latter relation for bosons and fermions. For the bosons is originates from the reality condition (5.2). The fermions $\Psi_{a\dot{\alpha}}$ and $\Psi^*_{a\dot{\alpha}}$, however, are independent. In this case it is the equations of motion which require $(b^{a\dot{\alpha}})^* = b^\dagger_{a\dot{\alpha}}$.

5.3 Tree-level S-matrix

We compute the (65536) components of the T-matrix as defined in (2.7) and (2.10), relying only on the manifest $\mathrm{SU}(2)^4$ symmetry. There are four kinds of particles that we can scatter: $Y_{a\dot{a}}$, $Z_{\alpha\dot{\alpha}}$, $\Psi_{a\dot{\alpha}}$, $\Upsilon_{\alpha\dot{a}}$. Let us consider the scattering two Y's. There are four different channels, which can be found by taking the tensor product of the corresponding representations, cf. table 1:

$$(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) \otimes (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) = (\mathbf{3},\mathbf{3},\mathbf{1},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) \,. \tag{5.14}$$

These four representations can be realized by the following states[12]

[12] The brackets { } and [ ] denote symmetrization and anti-symmetrization of two undotted or two dotted indices. The prime is written to distinguish different particle momenta.

$(\mathbf{3},\mathbf{3},\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
$\lvert Y_{\{a\{\dot{a}}Y'_{b\}\dot{b}\}}\rangle$	$\lvert Y_{\{a[\dot{a}}Y'_{b\}\dot{b}]}\rangle$	$\lvert Y_{[a\{\dot{a}}Y'_{b]\dot{b}\}}\rangle$	$\lvert Y_{[a[\dot{a}}Y'_{b]\dot{b}]}\rangle$
	$\lvert \Psi_{\{a[\dot{a}}\Psi'_{b\}\dot{\beta}]}\rangle$	$\lvert \Upsilon_{[\alpha\{\dot{a}}\Upsilon'_{\beta]\dot{b}\}}\rangle$	$\lvert \Psi_{[a[\dot{a}}\Psi'_{b]\dot{\beta}]}\rangle$
			$\lvert \Upsilon_{[\alpha[\dot{a}}\Upsilon'_{\beta]\dot{b}]}\rangle$
			$\lvert Z_{[\alpha[\dot{\alpha}}Z'_{\beta]\dot{\beta}]}\rangle$

Hence the action of the T-matrix is of the form

$$
\begin{aligned}
\mathbb{T}|Y_{a\dot{a}}Y'_{b\dot{b}}\rangle = &+\#\,|Y_{\{a\{\dot{a}}Y'_{b\}\dot{b}\}}\rangle + \#\,|Y_{\{a[\dot{a}}Y'_{b\}\dot{b}]}\rangle + \#\,|Y_{[a\{\dot{a}}Y'_{b]\dot{b}\}}\rangle + \#\,|Y_{[a[\dot{a}}Y'_{b]\dot{b}]}\rangle \\
&+\#\,\tfrac{1}{2}\epsilon_{\dot{a}\dot{b}}\epsilon^{\dot{\alpha}\dot{\beta}}|\Psi_{\{a\,\dot{\alpha}}\Psi'_{b\}\dot{\beta}}\rangle + \#\,\tfrac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}|\Upsilon_{\alpha\{\dot{a}}\Upsilon'_{\beta\,\dot{b}\}}\rangle \\
&+\#\,\tfrac{1}{2}\epsilon_{\dot{a}\dot{b}}\epsilon^{\dot{\alpha}\dot{\beta}}|\Psi_{[a\,\dot{\alpha}}\Psi'_{b]\dot{\beta}}\rangle + \#\,\tfrac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}|\Upsilon_{\alpha[\dot{a}}\Upsilon'_{\beta\,\dot{b}]}\rangle \\
&+\#\,\tfrac{1}{4}\epsilon_{\dot{a}\dot{b}}\epsilon_{ab}\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}|Z_{\alpha\dot{\alpha}}Z'_{\beta\dot{\beta}}\rangle\ .
\end{aligned}
\tag{5.15}
$$

The explicit computation yields

$$
\begin{aligned}
\mathbb{T}|Y_{a\dot{a}}Y'_{b\dot{b}}\rangle = \ \frac{1}{\varepsilon' p - \varepsilon p'}\Big[&\tfrac{1}{2}\big((p-p')^2 + 4pp'\big)|Y_{\{a\{\dot{a}}Y'_{b\}\dot{b}\}}\rangle \\
&+\tfrac{1}{2}(p-p')^2\big(|Y_{\{a[\dot{a}}Y'_{b\}\dot{b}]}\rangle + |Y_{[a\{\dot{a}}Y'_{b]\dot{b}\}}\rangle\big) \\
&+\tfrac{1}{2}\big((p-p')^2 - 4pp'\big)|Y_{[a[\dot{a}}Y'_{b]\dot{b}]}\rangle \\
&-2pp'\sinh\tfrac{\theta-\theta'}{2}\big(\tfrac{1}{2}\epsilon_{\dot{a}\dot{b}}\epsilon^{\dot{\alpha}\dot{\beta}}|\Psi_{\{a\,\dot{\alpha}}\Psi'_{b\}\dot{\beta}}\rangle + \tfrac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}|\Upsilon_{\alpha\{\dot{a}}\Upsilon'_{\beta\,\dot{b}\}}\rangle \\
&\qquad + \tfrac{1}{2}\epsilon_{\dot{a}\dot{b}}\epsilon^{\dot{\alpha}\dot{\beta}}|\Psi_{[a\,\dot{\alpha}}\Psi'_{b]\dot{\beta}}\rangle + \tfrac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}|\Upsilon_{\alpha[\dot{a}}\Upsilon'_{\beta\,\dot{b}]}\rangle\big)\Big]\ ,
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\mathbb{T}|Y_{a\dot{a}}Y'_{b\dot{b}}\rangle = &\ \frac{1}{2}\,\frac{(p-p')^2}{\varepsilon' p - \varepsilon p'}\,|Y_{a\dot{a}}Y'_{b\dot{b}}\rangle + \frac{pp'}{\varepsilon' p - \varepsilon p'}\,\big(|Y_{a\dot{b}}Y'_{b\dot{a}}\rangle + |Y_{b\dot{a}}Y'_{a\dot{b}}\rangle\big) \\
&- \frac{pp'}{\varepsilon' p - \varepsilon p'}\,\sinh\frac{\theta-\theta'}{2}\,\big(\epsilon_{\dot{a}\dot{b}}\epsilon^{\dot{\alpha}\dot{\beta}}|\Psi_{a\dot{\alpha}}\Psi'_{b\dot{\beta}}\rangle + \epsilon_{ab}\epsilon^{\alpha\beta}|\Upsilon_{\alpha\dot{a}}\Upsilon'_{\beta\dot{b}}\rangle\big)\ .
\end{aligned}
\tag{5.16}
$$

The bosonic part reproduces, of course, the computation in section 4 for $a=\frac{1}{2}$. Notice that the coefficients are such that the states on the right hand side do not differ in both undotted and dotted indices from the in-state on the left hand side. Such terms would prevent group-factorization (2.11) of the T-matrix.

The other components of $\mathbb{T}$ are listed in appendix D. The entire result can be written concisely by giving the coefficient functions as defined in (2.12) for one $\mathfrak{psu}(2|2)$ factor. We find[13]

$$\begin{aligned}
\mathrm{A}(p,p') &= -\mathrm{D}(p,p') = \frac{1}{4}\,\frac{(p-p')^2}{\varepsilon' p - \varepsilon p'}\,,\\
\mathrm{B}(p,p') &= -\mathrm{E}(p,p') = \frac{pp'}{\varepsilon' p - \varepsilon p'}\,,\\
\mathrm{C}(p,p') &= +\mathrm{F}(p,p') = -\frac{pp'}{\varepsilon' p - \varepsilon p'}\,\sinh\frac{\theta-\theta'}{2}\,,\\
\mathrm{G}(p,p') &= -\mathrm{L}(p,p') = -\frac{1}{4}\,\frac{p^2-p'^2}{\varepsilon' p - \varepsilon p'}\,,\\
\mathrm{H}(p,p') &= +\mathrm{K}(p,p') = \frac{pp'}{\varepsilon' p - \varepsilon p'}\,\cosh\frac{\theta-\theta'}{2}\,.
\end{aligned} \tag{5.17}$$

In order to compare with the form in section 2, note the following kinematical identities

$$\frac{1}{2}\sqrt{(\varepsilon+1)\,(\varepsilon'+1)}\,\big(\varepsilon' p - \varepsilon p' + p' - p\big) = -pp'\sinh\tfrac{\theta-\theta'}{2}\,, \tag{5.18}$$

$$\frac{1}{2}\,\frac{(\varepsilon+1)\,(\varepsilon'+1) - pp'}{\sqrt{(\varepsilon+1)\,(\varepsilon'+1)}} = \cosh\tfrac{\theta-\theta'}{2}\,. \tag{5.19}$$

5.4 Symmetries

The string states are constructed from oscillators that individually do not satisfy the level-matching condition (i.e. they carry nonvanishing worldsheet momentum). In this sense they can be considered off-shell. The symmetry algebra in the absence of the level-matching constraint in the uniform light-cone gauge-fixed theory is $\mathfrak{psu}(2|2)_L \times \mathfrak{psu}(2|2)_R \ltimes \mathbb{R}^2$ [30]. The total momentum is measured by an operator $\mathfrak{P}$ which appears as one of the central generators of the symmetry algebra. The other central generator is the total energy $\mathfrak{H}$.

In [30] the symmetry generators were found in term of the worldsheet fields. We would like to act with the symmetry generators on the scattering states; consequently, we need to know the oscillator representation of the symmetry generators. Since the nonlocal nature of the symmetry generators has already been taken into account, it suffices to focus on their local part. In the notation of section 3 this corresponds to analyzing the currents generically denoted by $\tilde{J}$ in equation (3.5). Computing the integrals along fixed-time slices, the oscillator representation of the generators of $\mathfrak{psu}(2|2)_L$ is (to quadratic order)

$$\begin{aligned}
\mathfrak{L}_a{}^b &= \int \frac{dp}{2\pi}\,\frac{1}{2}\Big[c^\dagger_{a\dot C}\,c^{b\dot C} - c^{\dagger b\dot C}\,c_{a\dot C}\Big]\,, &\quad \mathfrak{Q}_\alpha{}^b &= \int \frac{dp}{2\pi}\,(-)^{[\dot C]}\Big[u\,c^\dagger_{\alpha\dot C}\,c^{b\dot C} - v\,c^{\dagger b\dot C}\,c_{\alpha\dot C}\Big]\,,\\
\mathfrak{R}_\alpha{}^\beta &= \int \frac{dp}{2\pi}\,\frac{1}{2}\Big[c^\dagger_{\alpha\dot C}\,c^{\beta\dot C} - c^{\dagger\beta\dot C}\,c_{\alpha\dot C}\Big]\,, &\quad \mathfrak{S}_a{}^\beta &= \int \frac{dp}{2\pi}\,(-)^{[\dot C]}\Big[u\,c^\dagger_{a\dot C}\,c^{\beta\dot C} - v\,c^{\dagger\beta\dot C}\,c_{a\dot C}\Big]\,,
\end{aligned}$$

the generators of $\mathfrak{psu}(2|2)_R$ are

$$\begin{aligned}
\dot{\mathfrak{L}}_{\dot a}{}^{\dot b} &= \int \frac{dp}{2\pi}\,\frac{1}{2}\Big[c^\dagger_{C\dot a}\,c^{C\dot b} - c^{\dagger C\dot b}\,c_{C\dot a}\Big]\,, &\quad \dot{\mathfrak{Q}}_{\dot\alpha}{}^{\dot b} &= \int \frac{dp}{2\pi}\,\Big[u\,c^\dagger_{C\dot\alpha}\,c^{C\dot b} - v\,c^{\dagger C\dot b}\,c_{C\dot\alpha}\Big]\,,\\
\dot{\mathfrak{R}}_{\dot\alpha}{}^{\dot\beta} &= \int \frac{dp}{2\pi}\,\frac{1}{2}\Big[c^\dagger_{C\dot\alpha}\,c^{C\dot\beta} - c^{\dagger C\dot\beta}\,c_{C\dot\alpha}\Big]\,, &\quad \dot{\mathfrak{S}}_{\dot a}{}^{\dot\beta} &= \int \frac{dp}{2\pi}\,\Big[u\,c^\dagger_{C\dot a}\,c^{C\dot\beta} - v\,c^{\dagger C\dot\beta}\,c_{C\dot a}\Big]\,,
\end{aligned}$$

[13]Notice that there arise signs when T acts across a fermionic index:
$\mathbb{T}|\Phi_{A\dot A}\Phi'_{B\dot B}\rangle = (-)^{[\dot A]([B]+[D])}\,|\Phi_{C\dot A}\Phi'_{D\dot B}\rangle\,\mathrm{T}_{AB}^{CD} + (-)^{[B]([\dot A]+[\dot C])}\,|\Phi_{A\dot C}\Phi'_{B\dot D}\rangle\,\mathrm{T}_{\dot A\dot B}^{\dot C\dot D}$.

and the two dentral charge generators generators read

$$\mathfrak{P} = \int \frac{dp}{2\pi}\, p\, c^\dagger_{A\dot{A}}\, c^{A\dot{A}} , \qquad\qquad \mathfrak{H} = \int \frac{dp}{2\pi}\, \varepsilon\, c^\dagger_{A\dot{A}}\, c^{A\dot{A}} .$$

As before we have

$$u = \cosh\tfrac{\theta}{2} \quad , \quad v = \sinh\tfrac{\theta}{2} \quad , \quad p = \sinh\theta \quad , \quad \varepsilon = \cosh\theta \, . \tag{5.20}$$

These formulas give the oscillator representation of the centrally extended $\mathfrak{psu}(2|2)^2$ algebra. Since $\mathfrak{Q}$ and $\mathfrak{S}$ transform as the components of a Lorentz spinor [48], one can see from this form that this representation is related to a representation of the non-centrally extended algebra ($\mathfrak{P} = 0$) by a Lorentz boost. Recall that the free worldsheet Lagrangian indeed possesses Lorentz invariance.

The algebra. The generators $\mathfrak{P}$ and $\mathfrak{H}$ are the two central charges corresponding to total momentum and total energy of a representation. The rotation generators $\mathfrak{R}_\alpha{}^\beta$ and $\mathfrak{L}_a{}^b$ act onto a generic generator $\mathfrak{J}$ canonically as

$$[\mathfrak{L}_a{}^b, \mathfrak{J}_c] = \delta^b_c\, \mathfrak{J}_a - \tfrac{1}{2}\delta^b_a\, \mathfrak{J}_c \, , \qquad [\mathfrak{L}_a{}^b, \mathfrak{J}^c] = -\delta^c_a\, \mathfrak{J}^b + \tfrac{1}{2}\delta^b_a\, \mathfrak{J}^c \, , \tag{5.21}$$

$$[\mathfrak{R}_\alpha{}^\beta, \mathfrak{J}_\gamma] = \delta^\beta_\gamma\, \mathfrak{J}_\alpha - \tfrac{1}{2}\delta^\beta_\alpha\, \mathfrak{J}_\gamma \, , \qquad [\mathfrak{R}_\alpha{}^\beta, \mathfrak{J}^\gamma] = -\delta^\gamma_\alpha\, \mathfrak{J}^\beta + \tfrac{1}{2}\delta^\beta_\alpha\, \mathfrak{J}^\gamma \, . \tag{5.22}$$

The supercharges satisfy

$$\{\mathfrak{Q}_\alpha{}^a, \mathfrak{Q}_\beta{}^b\} = -\tfrac{1}{2}\epsilon_{\alpha\beta}\epsilon^{ab}\, \mathfrak{P} \, , \tag{5.23}$$

$$\{\mathfrak{S}_a{}^\alpha, \mathfrak{S}_b{}^\beta\} = -\tfrac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}\, \mathfrak{P} \, , \tag{5.24}$$

$$\{\mathfrak{Q}_\alpha{}^a, \mathfrak{S}_b{}^\beta\} = \delta^\beta_\alpha\, \mathfrak{L}_b{}^a + \delta^a_b\, \mathfrak{R}_\alpha{}^\beta + \tfrac{1}{2}\delta^\beta_\alpha\delta^a_b\, \mathfrak{H} \, . \tag{5.25}$$

Here we have the same central charge appearing in the anticommutator of $\mathfrak{Q}$ and $\mathfrak{S}$ with itself. This is due to the quadratic approximation made here. Including higher orders, one would find that the two central charges (both denoted here by $\mathfrak{P}$) are the complex conjugate of each other. The generators of $\mathfrak{psu}(2|2)_R$ obey identical algebra relations.

Single excitation representation. If we act with the supercharges defined above on a single oscillator, we find

$$\dot{\mathfrak{L}}_{\dot a}{}^{\dot b}|c^\dagger_{\dot c}\rangle = \delta^{\dot b}_{\dot c}|c^\dagger_{\dot a}\rangle - \tfrac{1}{2}\delta^{\dot b}_{\dot a}|c^\dagger_{\dot c}\rangle \, , \qquad \dot{\mathfrak{R}}_{\dot\alpha}{}^{\dot\beta}|c^\dagger_{\dot\gamma}\rangle = \delta^{\dot\beta}_{\dot\gamma}|c^\dagger_{\dot\alpha}\rangle - \tfrac{1}{2}\delta^{\dot\beta}_{\dot\alpha}|c^\dagger_{\dot\gamma}\rangle \, , \tag{5.26}$$

$$\dot{\mathfrak{Q}}_{\dot\alpha}{}^{\dot b}|c^\dagger_{\dot c}\rangle = u\, \delta^{\dot b}_{\dot c}|c^\dagger_{\dot\alpha}\rangle \, , \qquad \dot{\mathfrak{Q}}_{\dot\alpha}{}^{\dot b}|c^\dagger_{\dot\gamma}\rangle = -v\, \epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot b\dot c}|c^\dagger_{\dot c}\rangle \, , \tag{5.27}$$

$$\dot{\mathfrak{S}}_{\dot a}{}^{\dot\beta}|c^\dagger_{\dot c}\rangle = -v\, \epsilon_{\dot a\dot b}\epsilon^{\dot\beta\dot\gamma}|c^\dagger_{\dot\gamma}\rangle \, , \qquad \dot{\mathfrak{S}}_{\dot a}{}^{\dot\beta}|c^\dagger_{\dot\gamma}\rangle = u\, \delta^{\dot\beta}_{\dot\gamma}|c^\dagger_{\dot a}\rangle \, , \tag{5.28}$$

where the undotted indices remain unaffected and have been suppressed. This is the same representation as for a single excitation of the spin chain [12]. Comparing the two cases, we see that the coefficients a, b, c, d of [12] take the values $a = d = u$ and $b = c = -v$. Note that indeed $ad - bc = 1$ (required by the closure of the algebra), $\frac{1}{2}(ad + bc) = \frac{1}{2}\varepsilon$ (first central charge) and $ab = cd = -\frac{1}{2}p$ (two more central charges).

Invariance of the T-matrix. With the ingredients described above we can now verify that the tree-level worldsheet S-matrix derived in section 5.3 is invariant under $\mathfrak{psu}(2|2)^2$ transformations with the the coproduct action (3.11):

$$[\mathfrak{Q}_\alpha{}^b \otimes \mathfrak{F} + \mathfrak{F} \otimes \mathfrak{Q}_\alpha{}^b, \mathrm{T}] = +(\tfrac{1}{2}\mathfrak{P}\mathfrak{F}) \otimes \mathfrak{Q}_\alpha{}^b - \mathfrak{Q}_\alpha{}^b \otimes (\tfrac{1}{2}\mathfrak{P}\mathfrak{F}) \tag{5.29}$$

$$[\mathfrak{S}_a{}^\beta \otimes \mathfrak{F} + \mathfrak{F} \otimes \mathfrak{S}_a{}^\beta, \mathrm{T}] = -(\tfrac{1}{2}\mathfrak{P}\mathfrak{F}) \otimes \mathfrak{S}_a{}^\beta + \mathfrak{S}_a{}^\beta \otimes (\tfrac{1}{2}\mathfrak{P}\mathfrak{F}) \tag{5.30}$$

where $\mathfrak{F}$ acts as $\mathfrak{F}|c^\dagger_A\rangle = (-)^{[A]}|c^\dagger_A\rangle$. These equations arise from the expansion of (3.13) at large 't Hooft coupling; the terms appearing on the right hand side of these equations are a direct consequence of the non-trivial co-product for the action of the supercharges on multi-excitation states , cf. section 3.

6. Comparison with SYM

As we have mentioned previously the S-matrix of planar $\mathcal{N} = 4$ SYM can be constructed by using the fact that a choice of ferromagnetic spin chain vacuum state breaks the full $\mathfrak{psu}(2,2|4)$ to its $\mathfrak{psu}(2|2)^2$ subgroup. However this novel spin chain is "dynamic" in the sense that the number of lattice sites can change. This induces two additional central charges shared by both factors of the symmetry group.

For the spin chain, the scattering processes $\phi_a\phi_b \to \psi_\alpha\psi_\beta$ and $\psi_\alpha\psi_\beta \to \phi_a\phi_b$ involve the creation or annihilation of a vacuum lattice site, denoted by $\mathcal{Z}^\pm$, and it is these insertions which give rise to the non-trivial coproduct for the global charges in the spin chain (see [49],[41]) while at the same time being responsible for the appearence of the central charges. The S-matrix of [12] for a single SU(2|2) sector involving the scalar fields, ϕ_a, and the fermions, ψ_α with $a, \alpha = 1, 2$, is uniquely determined up to an overall phase by demanding that the S-matrix is invariant under the action of the $\mathfrak{su}(2|2)$ algebra where the generators act on the fields as:

$$\mathfrak{L}_a{}^b|\phi_c\rangle = \delta^b_c|\phi_a\rangle - \tfrac{1}{2}\delta^b_a|\phi_c\rangle \,, \qquad \mathfrak{R}_\alpha{}^\beta|\psi_\gamma\rangle = \delta^\beta_\gamma|\psi_\alpha\rangle - \tfrac{1}{2}\delta^\beta_\alpha|\psi_\gamma\rangle \,, \tag{6.1}$$

$$\mathfrak{Q}_\alpha{}^b|\phi_c\rangle = a\,\delta^b_c|\psi_\alpha\rangle \,, \qquad \mathfrak{Q}_\alpha{}^b|\psi_\gamma\rangle = b\,\epsilon_{\alpha\beta}\epsilon^{bc}|\phi_c\mathcal{Z}^+\rangle \,, \tag{6.2}$$

$$\mathfrak{S}_a{}^\beta|\phi_c\rangle = c\,\epsilon_{ab}\epsilon^{\beta\gamma}|\psi_\gamma\mathcal{Z}^-\rangle \,, \qquad \mathfrak{S}_a{}^\beta|\psi_\gamma\rangle = d\,\delta^\beta_\gamma|\phi_a\rangle \,; \tag{6.3}$$

and the extra central charges $\mathfrak{P}$ and $\mathfrak{K}$ required by the presence of the length-changing operators $\mathcal{Z}^\pm$ act as

$$\mathfrak{P}|\chi\rangle = ab\,|\chi\mathcal{Z}^+\rangle \quad , \quad \mathfrak{K}|\chi\rangle = cd\,|\chi\mathcal{Z}^-\rangle \,. \tag{6.4}$$

When these generators act on multi-particle states the $\mathcal{Z}^\pm$ operators introduce additional momentum-dependent phases which promote this algebra to a Hopf subalgebra.

The resulting S-matrix is given by

$$\mathbf{S}^B|\phi_a\phi'_b\rangle = \mathbf{A}^B|\phi'_{\{a}\phi_{b\}}\rangle + \mathbf{B}^B|\phi'_{[a}\phi_{b]}\rangle + \tfrac{1}{2}\mathbf{C}^B\varepsilon_{ab}\varepsilon^{\alpha\beta}|\psi'_\alpha\psi_\beta\mathcal{Z}^-\rangle \,, \tag{6.5}$$

$$\mathbf{S}^B|\psi_\alpha\psi'_\beta\rangle = \mathbf{D}^B|\psi'_{\{\alpha}\psi_{\beta\}}\rangle + \mathbf{E}^B|\psi'_{[\alpha}\psi_{\beta]}\rangle + \tfrac{1}{2}\mathbf{F}^B\varepsilon_{\alpha\beta}\varepsilon^{ab}|\phi'_a\phi_b\mathcal{Z}^+\rangle \,, \tag{6.6}$$

$$\mathbf{S}^B|\phi_a\psi'_\beta\rangle = \mathbf{G}^B|\psi'_\beta\phi_a\rangle + \mathbf{H}^B|\phi'_a\psi_\beta\rangle \,, \tag{6.7}$$

$$\mathbf{S}^B|\psi_\alpha\phi'_b\rangle = \mathbf{K}^B|\psi'_\alpha\phi_b\rangle + \mathbf{L}^B|\phi'_b\psi_\alpha\rangle \,, \tag{6.8}$$

with the coefficients

$$
\begin{aligned}
\mathbf{A}^B &= S^0_{pp'}\,\frac{x^+_{p'}-x^-_p}{x^-_{p'}-x^+_p}\,,\\
\mathbf{B}^B &= S^0_{pp'}\,\frac{x^+_{p'}-x^-_p}{x^-_{p'}-x^+_p}\left(1-2\,\frac{1-\frac{1}{x^+_p x^-_{p'}}}{1-\frac{1}{x^+_p x^+_{p'}}}\,\frac{x^-_{p'}-x^-_p}{x^+_{p'}-x^-_p}\right)\,,\\
\mathbf{C}^B &= S^0_{pp'}\,\frac{2\gamma_p\gamma_{p'}}{x^+_p x^+_{p'}}\,\frac{1}{1-\frac{1}{x^+_p x^+_{p'}}}\,\frac{x^-_{p'}-x^-_p}{x^-_{p'}-x^+_p}\,,\\
\mathbf{D}^B &= -S^0_{pp'},\\
\mathbf{E}^B &= -S^0_{pp'}\left(1-2\,\frac{1-\frac{1}{x^-_p x^+_{p'}}}{1-\frac{1}{x^-_p x^-_{p'}}}\,\frac{x^+_{p'}-x^+_p}{x^-_{p'}-x^+_p}\right)\,,\\
\mathbf{F}^B &= -S^0_{pp'}\,\frac{2}{\gamma_p\gamma_{p'}x^-_p x^-_{p'}}\,\frac{(x^+_p-x^-_p)(x^+_{p'}-x^-_{p'})}{1-\frac{1}{x^-_p x^-_{p'}}}\,\frac{x^+_{p'}-x^+_p}{x^-_{p'}-x^+_p}\,,\\
\mathbf{G}^B &= S^0_{pp'}\,\frac{x^+_{p'}-x^+_p}{x^-_{p'}-x^+_p}\,, \qquad\qquad \mathbf{H}^B = S^0_{pp'}\,\frac{\gamma_p}{\gamma_{p'}}\,\frac{x^+_{p'}-x^-_{p'}}{x^-_{p'}-x^+_p}\,,\\
\mathbf{K}^B &= S^0_{pp'}\,\frac{\gamma_{p'}}{\gamma_p}\,\frac{x^+_p-x^-_p}{x^-_{p'}-x^+_p}\,, \qquad\qquad \mathbf{L}^B = S^0_{pp'}\,\frac{x^-_{p'}-x^-_p}{x^-_{p'}-x^+_p}\,,
\end{aligned}
\tag{6.9}
$$

where

$$\gamma_p = |x^-_p - x^+_p|^{1/2} \tag{6.10}$$

and

$$x^\pm_p = \frac{\pi e^{\pm\frac{i}{2}p}}{\sqrt{\lambda}\sin\frac{p}{2}}\left(1+\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}\right)\,. \tag{6.11}$$

As mentioned before, the phase factor S^0 is undetermined by the algebraic construction. The one that correctly reproduces the semiclassical string spectrum via Bethe equations [3] is

$$S^0_{pp'} = \frac{1-\frac{1}{x^+_{p'}x^-_p}}{1-\frac{1}{x^-_{p'}x^+_p}}\,e^{i\theta(p,p')} \tag{6.12}$$

with the dressing phase given by [15] (to the leading order in $1/\sqrt{\lambda}$)

$$
\begin{aligned}
\theta(p,p') &= \frac{\sqrt{\lambda}}{2\pi}\sum_{r,s=\pm} rs\,\chi(x^r_p, x^s_{p'}),\\
\chi(x,y) &= (x-y)\left[\frac{1}{xy}+\left(1-\frac{1}{xy}\right)\ln\left(1-\frac{1}{xy}\right)\right]\,.
\end{aligned}
\tag{6.13}
$$

In the comparison with the worldsheet calculation we are actually interested in the coefficients of $\mathcal{P}_{\rm g}P^u_{pp'}S^B$, where $\mathcal{P}_{\rm g}$ is the graded permutation operator and $P^u_{pp'}$ exchanges the excitation momenta. Furthermore, in order to find the S-matrix for the full PSU$(2,2|4)$ theory we use the relation[14],

$$\$ = \frac{1}{\mathbf{A}^B}\mathbf{S}^B \otimes \mathbf{S}^B \quad , \quad \$^{C\dot{C}D\dot{D}}_{A\dot{A}B\dot{B}}(p,p') = \frac{1}{\mathbf{A}^B}(\mathbf{S}^B)^{CD}_{AB}(p,p')(\mathbf{S}^B)^{\dot{C}\dot{D}}_{\dot{A}\dot{B}}(p,p')\ . \tag{6.14}$$

Consequently we can relate the above coefficients to those of $\mathbf{S}$ used in section 2

$$\begin{aligned}
\mathbf{A} &= \tfrac{1}{2\sqrt{\mathbf{A}^B}}(\mathbf{A}^B - \mathbf{B}^B)\ , & \mathbf{B} &= \tfrac{1}{2\sqrt{\mathbf{A}^B}}(\mathbf{A}^B + \mathbf{B}^B)\ , & \mathbf{C} &= \tfrac{1}{\sqrt{\mathbf{A}^B}}\mathbf{C}^B\ ,\\
\mathbf{D} &= \tfrac{1}{2\sqrt{\mathbf{A}^B}}(-\mathbf{D}^B + \mathbf{E}^B)\ , & \mathbf{E} &= \tfrac{1}{2\sqrt{\mathbf{A}^B}}(-\mathbf{D}^B - \mathbf{E}^B)\ , & \mathbf{F} &= -\tfrac{1}{\sqrt{\mathbf{A}^B}}\mathbf{F}^B\ ,\\
\mathbf{G} &= \tfrac{1}{\sqrt{\mathbf{A}^B}}\mathbf{G}^B\ , & \mathbf{H} &= \tfrac{1}{\sqrt{\mathbf{A}^B}}\mathbf{H}^B\ ,\\
\mathbf{L} &= \tfrac{1}{\sqrt{\mathbf{A}^B}}\mathbf{L}^B\ , & \mathbf{K} &= \tfrac{1}{\sqrt{\mathbf{A}^B}}\mathbf{K}^B\ .
\end{aligned} \tag{6.15}$$

In order to compare our worldsheet results with those of the spin chain S-matrix of [12] we must expand the latter in $1/\sqrt{\lambda}$. But first we should understand how the spin chain momenta are related to the worldsheet momenta. As part of the gauge fixing of the worldsheet theory we chose the density of the light-cone momentum to be a constant which in turn fixed the string length to be $\mathcal{J} = 2\pi J_+/\sqrt{\lambda}$ where J_+ is the momentum. Then, we took $\mathcal{J}$ to be infinite, which allowed for a sensible definition of the S-matrix, and expanded in powers of $\frac{2\pi}{\sqrt{\lambda}}$ which acts as a loop counting parameter. This should be contrasted with the spin chain whose length L is identified with the momentum J plus an additional term that depends on the number of excitations[15]: $L = J + M$. Going from the spin chain to the string worldsheet involves the rescaling by a factor of $\sqrt{\lambda}/2\pi$, which affects all dimensional quantities and in particular all momenta in (6.9), which should be rescaled as

$$p \longrightarrow \frac{2\pi p}{\sqrt{\lambda}} \qquad p_{\text{chain}} = \frac{2\pi}{\sqrt{\lambda}} p_{\text{string}}\ . \tag{6.16}$$

Indeed, once we impose the periodic boundary conditions, the spin chain momentum is quantized in the units of $2\pi/J$, while in the sigma-model the momentum quantization unit is $\sqrt{\lambda}/J$. We should therefore first rescale as above all momenta in the spin chain S-matrix and then expand it in $1/\sqrt{\lambda}$. The matrix elements in (6.9) depend on $1/\sqrt{\lambda}$ only through $x_p^\pm$. Therefore, the strong-coupling expansion is equivalent to the low-momentum expansion of the spin chain S-matrix. For the kinematical variables (6.11) the rescaling of momenta yields:

$$x_p^\pm = \frac{1+\varepsilon}{p}\left(1 \pm \frac{i\pi p}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right)\right)\ . \tag{6.17}$$

Note that in the limit we are considering here all information about bound states appears at higher orders in the $1/\sqrt{\lambda}$ expansion.

[14] The full S-matrix has to be divided by A^B, because the $\mathfrak{psu}(2|2)$ S-matrix was defined in [12] as the physical scattering matrix of the fields $\Phi_{A\mathrm{i}}$. In addition to $\mathbf{S}$ for the left $\mathfrak{psu}(2|2)$ indices, the scattering of this field receives contribution from $\mathbf{S}^{\mathrm{i}\,\mathrm{i}}_{\mathrm{i}\,\mathrm{i}} = A^B$.

[15] Various excitations contribute differently to the length, see [6] for the precise definition. For the sake of our argument, it is enough to known that $J \to \infty$ and $M = O(1)$ in the decompactification limit. The difference between L and J then becomes negligible.

The expansion of (6.9)–(6.13) in $1/\sqrt{\lambda}$ is a tedious but straightforward exercise. The small-momentum expansion of the dressing phase (6.13) was computed in [22]:

$$\begin{aligned}\theta(p,p') &= -\frac{2\pi}{\sqrt{\lambda}}\,(1+\varepsilon)\,\big(1+\varepsilon'\big)\,\left.\frac{\partial^2\chi}{\partial x\,\partial y}\right|_{x=\frac{1+\varepsilon}{p}\,,\;y=\frac{1+\varepsilon'}{p'}} \\ &= \frac{2\pi}{\sqrt{\lambda}}\,\frac{\frac{1}{2}\,(p-p')^2-(p-p')(\varepsilon' p-\varepsilon p')+\frac{1}{2}(\varepsilon' p-\varepsilon p')^2}{\varepsilon' p-\varepsilon p'}\,. \end{aligned}\tag{6.18}$$

Expanding the matrix elements we find for the components (2.12) of the T-matrix:

$$\begin{aligned}
\mathrm{A}(p,p') &= \frac{1}{4}\Big[\big(\varepsilon' p-\varepsilon p'\big)-2(p-p')+\frac{(p-p')^2}{\varepsilon' p-\varepsilon p'}\Big]\,,\\
\mathrm{B}(p,p') &= -\mathrm{E}(p,p') = \frac{pp'}{\varepsilon' p-\varepsilon p'}\,,\\
\mathrm{C}(p,p') &= \mathrm{F}(p,p') = \frac{1}{2}\frac{\sqrt{(\varepsilon+1)\,(\varepsilon'+1)}\,(\varepsilon' p-\varepsilon p'+p'-p)}{\varepsilon' p-\varepsilon p'}\,,\\
\mathrm{D}(p,p') &= \frac{1}{4}\Big[\big(\varepsilon' p-\varepsilon p'\big)-\frac{(p-p')^2}{\varepsilon' p-\varepsilon p'}\Big]\,,\\
\mathrm{G}(p,p') &= -\mathrm{L}(p',p) = \frac{1}{4}\Big[\big(\varepsilon' p-\varepsilon p'\big)+2p'-\frac{p^2-p'^2}{\varepsilon' p-\varepsilon p'}\Big]\,,\\
\mathrm{H}(p,p') &= \mathrm{K}(p,p') = \frac{1}{2}\,\frac{pp'}{\varepsilon' p-\varepsilon p'}\,\frac{(\varepsilon+1)\,(\varepsilon'+1)-pp'}{\sqrt{(\varepsilon+1)\,(\varepsilon'+1)}}\,.
\end{aligned}\tag{6.19}$$

This should be compared with the string calculation in the constant-J gauge given in (2.13) for $a=0$. We note that the results almost agree, the only difference being terms which are linear in the momenta. These terms should not affect the physical spectrum when the S-matrix is plugged into the asymptotic Bethe equations. We suspect that when the linear terms are promoted to the linear phases in the exact S-matrix, they account for the difference between the length of the spin chain and the internal length of the string. This can be seen most clearly in the rank-one sectors; for example, the bosonic $\mathfrak{su}(2)$ sector of the spin chain is described by the Bethe equation

$$e^{iLp_i} = \prod_{j\neq i}^{M} \mathbf{S}^B_{\mathfrak{su}(2)}(p_i,p_j).\tag{6.20}$$

The string length is proportional to the R-charge J, but for the spin chain the length is $L=J+M$, where M is the number of impurities. In order to compare with the string theory we must rewrite the equation as

$$e^{iJp_i} = \prod_{j\neq i}^{M} \mathbf{S}^B_{\mathfrak{su}(2)}(p_i,p_j)e^{i(p_j-p_i)}.\tag{6.21}$$

Thus there are new terms in the S-matrix, which after the rescaling described above, contribute terms linear in momentum at order $1/\sqrt{\lambda}$. The appropriate change in length is different for the various impurities and one should carefully trace through the effects in the Bethe equations to see exactly how the string and spin chain S-matrices are related, which is beyond the scope of the present paper.

If, instead of (6.12), we take

$$S^0_{pp'} = e^{i\frac{p-p'}{2}}, \tag{6.22}$$

we find the resulting T-matrix is given by

$$\begin{aligned}
\mathrm{A}(p,p') &= \frac{1}{4}\frac{(p-p')^2}{\varepsilon' p - \varepsilon p'}\,,\\
\mathrm{B}(p,p') &= -\mathrm{E}(p,p') = \frac{pp'}{\varepsilon' p - \varepsilon p'}\,,\\
\mathrm{C}(p,p') &= \mathrm{F}(p,p') = \frac{1}{2}\frac{\sqrt{(\varepsilon+1)\,(\varepsilon'+1)}\,(\varepsilon' p - \varepsilon p' + p' - p)}{\varepsilon' p - \varepsilon p'}\,,\\
\mathrm{D}(p,p') &= \frac{1}{4}\left[2(p-p') - \frac{(p-p')^2}{\varepsilon' p - \varepsilon p'}\right],\\
\mathrm{G}(p,p') &= -\mathrm{L}(p',p) = \frac{1}{4}\left[2p' - \frac{p^2 - p'^2}{\varepsilon' p - \varepsilon p'}\right],\\
\mathrm{H}(p,p') &= \mathrm{K}(p,p') = \frac{1}{2}\frac{pp'}{\varepsilon' p - \varepsilon p'}\frac{(\varepsilon+1)\,(\varepsilon'+1) - pp'}{\sqrt{(\varepsilon+1)\,(\varepsilon'+1)}}\,.
\end{aligned} \tag{6.23}$$

This agrees with the string calculation in the light-cone gauge ($a = 1/2$), again up to terms linear in momenta.

7. Conclusions and discussion

The gauge-fixed sigma-model in $\mathrm{AdS}_5 \times S^5$ is a rather complicated 2d quantum field theory. Even at tree level, the calculations of the finite-size spectrum in [27] and of the S-matrix here involve complicated combinatorics. We have analyzed the latter calculation in detail and explicitly shown that the scattering matrix has all the required factorization properties consistent with integrability. A crucial ingredient was the fact that the action of the symmetry algebra on multi-particle states is given by a nontrivial coproduct.

A similar (albeit not identical) nonstandard realization of the symmetry algebra occurs on the gauge theory side of the duality [12]. Though different, these nonstandard realizations are crucial for the positive comparison of the worldsheet and the spin chain picture of the dual SYM theory. The difference we noted between the realization of symmetries can therefore be identified as a gauge degree of freedom. Indeed, there appears to exist a nonlocal field redefinition [40] that relates the two coproducts.

It would be very interesting to extend our calculations to include loop effects. Such a calculation would provide further nontrivial checks on standing conjectures regarding the S-matrices appearing in the context of the AdS/CFT correspondence. As mentioned above, at least one nonlocal field redefinition is necessary to directly identify at the classical level the fields and the algebraic structures on the two sides of the duality. Such redefinitions have the potential of altering the quantum theories. It is thus not immediately obvious which worldsheet one should use for computing quantum corrections to the scattering

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matrix. A possible guide is provided by the symmetry algebra described in section 3. There we argued that perturbative quantum corrections cannot alter the coproduct derived classically. Imposing this as a constraint on the definition of the quantum theory may uniquely identify it. Higher loop corrections in this theory should reproduce the higher order terms in the $1/\sqrt{\lambda}$ expansion of the spin chain S-matrix described in section section 6.

Two important issues that we have not discussed here are crossing symmetry and analyticity of the S-matrix. While the former is a kinematical restriction, the analytic properties of the S-matrix contain information about the spectrum of bound states of the theory. In the bootstrap approach to integrable relativistic quantum field theories these properties are very important, along with the quantum Yang-Baxter equation, in determining the S-matrix (up to a smooth phase) and the spectrum [47].

The AdS/CFT sigma-model in the light-cone (or any other unitary) gauge lacks relativistic invariance. This is reflected in the structure of the S-matrix, which depends on the individual momenta of the incoming particles rather than on their particular combination such as the difference of rapidities in relativistic theories. Nevertheless, based on algebraic considerations, the S-matrix of AdS/CFT was conjectured to satisfy a crossing relation [14]. Recalling that in relativistic quantum field theory the crossing symmetry is a simple consequence of the Feynman rules [50] and noting that two-dimensional Lorentz invariance is only broken spontaneously on the worldsheet, it is not unnatural to hope that diagrammatic perturbation theory of the type used in this paper may be helpful in understanding the behavior of the S-matrix under particle-antiparticle transformation. A different perspective on the connection with the crossing symmetry of relativistic field theories may be provided by the construction of [51].

Comparatively little is known about the analytic properties of the S-matrix. According to the standard bootstrap philosophy, bound states of the theory correspond to (complex) simple poles of the scattering matrix[16]. While the physical meaning of some higher order poles is known, an interpretation of more serious singularities of a two-dimensional S-matrix is currently lacking. Our tree-level calculation of the S-matrix does not yield direct information on its analytic structure. In particular, the two-magnon bound state present in the gauge theory spin chain exhibits, in our expansion, a difference of order $i/\sqrt{\lambda}$ between the corresponding world sheet momenta of its constituents; thus, it must be a quantum effect and should be accessible only after the perturbative series is (partly) resumed. An efficient way to gain access to the analytic structure of the S-matrix at the classical level is the analysis of the scattering of worldsheet solitons. In the limit of small charges and small momenta, the $1/\sqrt{\lambda}$ expansion of the soliton S-matrix reduces – in the appropriate gauge – to the results of perturbative calculations of the type described here.

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[16]The reverse does not always hold and simple poles do not always correspond to bound states.

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A. Symmetry considerations

As mentioned in section 2 the gauge-invariant worldsheet theory as well as the worldsheet theory in conformal gauge are classically integrable. Formally, one may think of gauge-fixing the diffeomorphism invariance as equivalent with expanding around some classical solution; for the light-cone gauge this is the BMN point-like string [52] combined with solving a subset of the classical equations of motion. It is therefore reasonable to expect that diffeomorphism-gauge-fixed worldsheet theory remains integrable. κ-symmetry gauge-fixing cannot be understood in a similar language; however, in explicit examples, it is possible to see that integrability is preserved.

Furthermore, there exist known examples in which despite the symmetry algebra being centrally-extended [30], the symmetry transformations act on multi-excitation states via the Leibniz rule. It is worth stressing that virtually in all known continuum integrable models this expectation is in fact realized and it relies only on the fact that in a quantum theory operators act via commutators.

Additionally, one may expect [42, 43] that the S-matrix is determined up to a phase by the symmetries of the theory, in particular the nonlocal symmetries and it again turns out that this expectation is realized in most existing integrable field theories.

A.1 Leibniz rule and symmetry constraints

Under the assumption that the global symmetries act following the Leibniz rule it is quite trivial to impose their conservation in the scattering process. For this purpose we need some sufficiently general action on single excitations. Denoting by $B^C(p)$ the creation operators, the symmetry generators $Q^A_{(1)B}$[17] act on these excitations as

$$\begin{aligned}\{Q_{(1)}{}^A{}_B, B^C(p)\} = \{&Q_{(1)}{}^A{}_B, B^C(p)\}_0 \\ &+ E_-^{AC} E_{+BD}\{c, B^D(p)\} + E_+^{AC} E_{-BD}\{c^*, B^D(p)\}\ .\end{aligned} \tag{A.1}$$

where $E_\pm$ are defined as

$$E_+ = \begin{pmatrix} \epsilon_{ab} & 0 \\ 0 & 0 \end{pmatrix}\ ,\ \ E_- = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_{\alpha\beta} \end{pmatrix}\ ,\ \ I_+ = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix}\ ,\ \ I_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{A.2}$$

and have we introduced $I_\pm$ for later convenience.

In the equation (A.1)

$$\{Q_{(1)}{}^A{}_B, B^C(p)\}_0 = f^{AC}_{BD}(p) B^D(p) \quad , \quad \{c, B^C(p)\}_0 = c(p) B^C(p) \tag{A.3}$$

[17] Here $Q_{(1)}$ uniformly covers both the bosonic and the fermionic $\mathfrak{psu}(2|2)$ generators.

represent the action of symmetry generators in the absence of the central extension and of the central charge, respectively. The coefficients $f^{AC}_{BD}(p)$ have manifest $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ symmetry and may be written as

$$\begin{aligned} f^{AC}_{BD}(p) = &\left(I_{+}{}^{C}_{B}I_{+}{}^{A}_{D} - \tfrac{1}{2}I_{+}{}^{A}_{B}I_{+}{}^{C}_{D}\right) + \left(I_{-}{}^{C}_{B}I_{-}{}^{A}_{D} - \tfrac{1}{2}I_{-}{}^{A}_{B}I_{-}{}^{C}_{D}\right) \\ &+ a(p)I_{-}{}^{A}_{D}I_{+}{}^{C}_{B} + d(p)I_{+}{}^{A}_{D}I_{-}{}^{C}_{B}\ . \end{aligned} \tag{A.4}$$

The functions $c(p)$, $a(p)$ and $d(p)$ are determined by the worldsheet sigma-model together with its Poisson structure. The momentum displayed as their argument is that of the excitation the generators act on.

Among the many consequences of the vanishing of the commutation of the S-matrix and the $\mathfrak{psu}(2|2)$ generators are

$$\begin{aligned} \mathbf{C}(p,p') =& \ \frac{1}{2a(p)}\Big[c(p')\mathbf{L}(p,p') - c(p)\mathbf{H}(p,p') - \big(\mathbf{D}(p,p') - \mathbf{E}(p,p')\big)c^*(p')\Big] \\ =& -\frac{1}{2a(p')}\Big[c(p')\mathbf{K}(p,p') - c(p)\mathbf{G}(p,p') + \big(\mathbf{D}(p,p') - \mathbf{E}(p,p')\big)c^*(p)\Big] \end{aligned} \tag{A.5}$$

$$\begin{aligned} \mathbf{F}(p,p') =& \ \frac{1}{2d(p)}\Big[c^*(p')\mathbf{G}(p,p') - c^*(p)\mathbf{K}(p,p') - \big(\mathbf{A}(p,p') - \mathbf{B}(p,p')\big)c(p')\Big] \\ =& -\frac{1}{2d(p')}\Big[c^*(p')\mathbf{H}(p,p') - c^*(p)\mathbf{L}(p,p') + \big(\mathbf{A}(p,p') - \mathbf{B}(p,p')\big)c(p)\Big]\ . \end{aligned} \tag{A.6}$$

The conservation of the first nonlocal charge provides further constraints on the scattering matrix which may be derived by considering the conservation of the first nonlocal charge in the scattering process. As we discuss in more detail in section 3, under the assumption that the $\mathfrak{psu}(2|2)$ generators act following the Leibniz rule, general considerations [53] imply that the action of the bilocal charge on 2-particle states is

$$\begin{aligned} {Q_{(2)}}^{S}{}_{T}|\Phi_A(p)\rangle\otimes|\Phi_B(p')\rangle = &\big({Q_{(2)}}^{S}{}_{T}|\Phi_A(p)\rangle\big)\otimes|\Phi_B(p')\rangle \\ &+|\Phi_A(p)\rangle\otimes\big({Q_{(2)}}^{S}{}_{T}|\Phi_B(p')\rangle\big) \\ &+\big({Q_{(1)}}^{S}{}_{M}|\Phi_A(p)\rangle\big)\otimes\big({Q_{(1)}}^{M}{}_{T}|\Phi_B(p')\rangle\big) \end{aligned} \tag{A.7}$$

It is somewhat less trivial to extract information that is not already included in the conservation of the global symmetry generators[18]. It is however easy to argue on general grounds that, in the presence of the central charges, the conservation of $Q_{(2)}$

$$\mathbf{S}Q_{(2)}|\Phi_A(p)\Phi_B(p')\rangle = Q_{(2)}\mathbf{S}|\Phi_A(p)\Phi_B(p')\rangle \tag{A.8}$$

that the $\mathbf{C}(p,p')$ and/or $\mathbf{F}(p,p')$ be nonvanishing.

Indeed, one may break the action of $Q_{(2)}$ into two parts, with even and odd parity in flavor space and similarly for the S-matrix:

$$Q_{(2)} = Q^{\rm even}_{(2)} + Q^{\rm odd}_{(2)} \quad , \quad \mathbf{S} = \mathbf{S}^{\rm even} + \mathbf{S}^{\rm odd}\ . \tag{A.9}$$

[18]This is so because it necessarily requires knowledge of the action of the bilocal charge on single-particle states.

The odd-parity component of (A.8)

$$[Q_{(2)}^{\text{odd}}, \mathbf{S}^{\text{even}}] + [Q_{(2)}^{\text{even}}, \mathbf{S}^{\text{odd}}] = 0 \tag{A.10}$$

is then an inhomogeneous linear equation for the unknown functions $\mathbf{C}(p, p')$ and $\mathbf{F}(p, p')$ with the inhomogeneous term provided by the central charges of the algebra. It is worth pointing out that the nontriviality of this equation arises from the fact that the structure functions (A.3) are momentum-dependent. This departs from previous analyses of the relation between the Yang-Baxter equation and nonlocal integrals of motion.

While this discussion was rather qualitative, it points to the possibility that the Lagrangian, the centrally-extended $\mathfrak{psu}(2|2)^2$ symmetry and existence of nonlocal charges have a chance of being consistent with each other in the context of the assumptions listed here. A more detailed analysis shows that to find exact agreement one must also include the effects of the non-trivial coproduct as was described in section 3.

B. Scattering of fermions in constant-J light-cone gauge

In this appendix we will consider the superstring in the constant-J light-cone gauge and show that up to terms linear in momenta the S-matrix is that of ref. [12] when we choose the overall phase factor to be that conjectured by AFS [15]. We start with the light-cone Hamiltonian described in [27], restrict to a SU(2|2) sector and calculate the S-matrix for this subsector. For the constant-J gauge we introduce the light-cone coordinates

$$x^+ = t \quad , \quad x^- = \phi - t \tag{B.1}$$

and fix the gauge,

$$x^+ = \tau \quad , \quad p_- = 1 \quad , \quad \Gamma^+\theta = 0 \tag{B.2}$$

where p_- is the light-cone momentum density, θ is a complex positive chirality spinor and Γ^A are the ten dimensional Dirac matrices. The light-cone Lagrangian is written in terms of the physical fields which are the eight bosons z^i, $i = 1, \ldots, 4$, $y^{i'}$, $i' = 5, \ldots, 8$ and the eight component spinors ψ and $\psi^\dagger$. The fermions further break into $\hat{\psi}$ and $\tilde{\psi}$ which are even or odd under the action of $\Pi = \gamma^1\gamma^2\gamma^3\gamma^4$ where the γ^i are the 8×8 γ-matrices:

$$\Pi\hat{\psi} = \hat{\psi}, \qquad \Pi\tilde{\psi} = -\tilde{\psi}. \tag{B.3}$$

The spinors $\hat{\psi}$ transform in the $(1, 2; 1, 2)$ of the $\mathrm{SU}(2)^4$, while $\tilde{\psi}$ transform as $(2, 1; 2, 1)$. In what follows we will restrict our attention to the $y^{i'}$ bosons and the $\tilde{\psi}$ fermions. The relevant part of the Lagrangian (dropping the tilde on the ψ) is $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$, where,

$$\mathcal{L}_0 = \frac{1}{2}\left(\dot{y}^2 - \acute{y}^2 - y^2\right) + i\psi^\dagger\dot{\psi} + \frac{i}{2}(\psi\acute{\psi} + \psi^\dagger\acute{\psi}^\dagger) + \psi^\dagger\psi \tag{B.4}$$

and $\mathcal{L}_{int} = \mathcal{L}_{BB} + \mathcal{L}_{FF} + \mathcal{L}_{BF}$ with

$$\mathcal{L}_{BB} = \frac{1}{\sqrt{\lambda}}\left[-\frac{1}{2}y^2\acute{y}^2 + \frac{1}{8}(y^2)^2 - \frac{1}{8}\left((\dot{y}^2)^2 + 2\dot{y}^2\acute{y}^2 + (\acute{y}^2)^2\right) + \frac{1}{2}(\acute{y}\cdot\dot{y})^2\right] \tag{B.5}$$

$$\mathcal{L}_{FF} = -\frac{1}{4\sqrt{\lambda}}\Bigg[-i\left[(\acute{\psi}\psi)+(\acute{\psi}^\dagger\psi^\dagger)\right](\psi^\dagger\psi)-\frac{1}{2}(\acute{\psi}\psi+\acute{\psi}^\dagger\psi^\dagger)^2 \\ +\frac{1}{2}\Big((\psi^\dagger\acute{\psi})-(\acute{\psi}^\dagger\psi)\Big)^2+\frac{i}{12}(\psi\gamma^{jk}\psi^\dagger)(\psi^\dagger\gamma^{jk}\acute{\psi}^\dagger) \\ +\frac{i}{48}(\psi\gamma^{jk}\psi+\psi^\dagger\gamma^{jk}\psi^\dagger)(\acute{\psi}^\dagger\gamma^{jk}\psi-\psi^\dagger\gamma^{jk}\acute{\psi})-(j,k\Leftrightarrow j'k')\Bigg] \tag{B.6}$$

$$\mathcal{L}_{BF} = \frac{1}{\sqrt{\lambda}}\Bigg[-\frac{i}{4}[\dot{y}^2+\acute{y}^2+y^2](\psi\acute{\psi}+\psi^\dagger\acute{\psi}^\dagger)-\frac{i}{2}(\dot{y}\cdot\acute{y})(\psi^\dagger\acute{\psi}+\psi\acute{\psi}^\dagger) \\ -\frac{1}{2}\acute{y}^2(\psi^\dagger\psi)-\frac{i}{4}(\acute{y}_{j'}y_{k'})(\psi\gamma^{j'k'}\psi+\psi^\dagger\gamma^{j'k'}\psi^\dagger) \\ +\frac{1}{4}(\dot{y}^{j'}\acute{y}^{k'})(\psi\gamma^{j'k'}\psi-\psi^\dagger\gamma^{j'k'}\psi^\dagger)\Bigg]. \tag{B.7}$$

To properly identify the SU(2|2) sector it is necessary to identify how the fields transform under the $\mathrm{SU}(2)^2$ symmetries. We will use the representation for the 8×8 γ-matrices

$$\begin{aligned} \gamma^1 &= \epsilon\times\epsilon\times\epsilon & \gamma^5 &= \tau_3\times\epsilon\times\mathbb{1} \\ \gamma^2 &= \mathbb{1}\times\tau_1\times\epsilon & \gamma^6 &= \epsilon\times\mathbb{1}\times\tau_1 \\ \gamma^3 &= \mathbb{1}\times\tau_3\times\epsilon & \gamma^7 &= \epsilon\times\mathbb{1}\times\tau_3 \\ \gamma^4 &= \tau_1\times\epsilon\times\mathbb{1} & \gamma^8 &= \mathbb{1}\times\mathbb{1}\times\mathbb{1} \end{aligned} \tag{B.8}$$

with

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{B.9}$$

The generators of the four SU(2) factors symmetry can be expressed as 8×8 SO(8) matrices:

$$\begin{aligned} \Sigma_1^\pm &= -\frac{1}{4i}(\gamma^2\gamma^3\pm\gamma^1\gamma^4) & \Omega_1^\pm &= \frac{1}{4i}(-\gamma^6\gamma^7\pm\gamma^8\gamma^5) \\ \Sigma_2^\pm &= -\frac{1}{4i}(\gamma^3\gamma^1\pm\gamma^2\gamma^4) & \Omega_2^\pm &= \frac{1}{4i}(-\gamma^7\gamma^5\pm\gamma^8\gamma^6) \\ \Sigma_3^\pm &= -\frac{1}{4i}(\gamma^1\gamma^2\pm\gamma^3\gamma^4) & \Omega_3^\pm &= \frac{1}{4i}(-\gamma^5\gamma^6\pm\gamma^8\gamma^7) . \end{aligned} \tag{B.10}$$

and we can rewrite the fermions $\tilde{\psi}$ in the $(2,1;2,1)$ representation in a notation closer to that used previously, e.g. section 5,

$$\tilde{\psi} = \frac{1}{2}\begin{pmatrix} -\Upsilon_{4\dot{2}}+\Upsilon_{3\dot{1}} \\ -i(\Upsilon_{3\dot{2}}-\Upsilon_{4\dot{1}}) \\ i(\Upsilon_{3\dot{2}}-\Upsilon_{4\dot{1}}) \\ -\Upsilon_{4\dot{2}}+\Upsilon_{3\dot{1}} \\ -i(\Upsilon_{4\dot{2}}+\Upsilon_{3\dot{1}}) \\ -\Upsilon_{3\dot{2}}-\Upsilon_{4\dot{1}} \\ -\Upsilon_{3\dot{2}}-\Upsilon_{4\dot{1}} \\ i(\Upsilon_{4\dot{2}}+\Upsilon_{3\dot{1}}) \end{pmatrix} \quad , \quad \tilde{\psi}^\dagger = \frac{1}{2}\begin{pmatrix} -\Upsilon^*_{4\dot{2}}+\Upsilon^*_{3\dot{1}} \\ -i(\Upsilon^*_{3\dot{2}}-\Upsilon^*_{4\dot{1}}) \\ i(\Upsilon^*_{3\dot{2}}-\Upsilon^*_{4\dot{1}}) \\ -\Upsilon^*_{4\dot{2}}+\Upsilon^*_{3\dot{1}} \\ -i(\Upsilon^*_{4\dot{2}}+\Upsilon^*_{3\dot{1}}) \\ -\Upsilon^*_{3\dot{2}}-\Upsilon^*_{4\dot{1}} \\ -\Upsilon^*_{3\dot{2}}-\Upsilon^*_{4\dot{1}} \\ i(\Upsilon^*_{4\dot{2}}+\Upsilon^*_{3\dot{1}}) \end{pmatrix} . \tag{B.11}$$

The $\Upsilon_{\alpha\dot{a}}$ transform non-trivially under the SU(2)'s generated by Σ^+, which acts on the undotted index, and Ω^+, which acts on the dotted index. We will be interested in the $\Upsilon_{\alpha\dot{1}}$ which have Ω_3^+ charge $-\frac{1}{2}$. Using the corresponding action of the SU(2) generators on the y bosons transforming as $(1,1;2,2)$:

$$
\begin{aligned}
\Omega_1^+ &= \frac{1}{2i}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} &\qquad \Omega_1^- &= \frac{1}{2i}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
\Omega_2^+ &= \frac{1}{2i}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} &\qquad \Omega_2^- &= \frac{1}{2i}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
\Omega_3^+ &= \frac{1}{2i}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} &\qquad \Omega_3^- &= \frac{1}{2i}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}
\end{aligned}
\tag{B.12}
$$

and we can introduce the following complex bosons:

$$
\begin{aligned}
Y^{1\dot{1}} &= \frac{1}{\sqrt{2}}\left(y^5 - iy^6\right) \ , &\qquad Y^{2\dot{2}} &= \frac{1}{\sqrt{2}}\left(y^5 + iy^6\right) \ , \\
Y^{1\dot{2}} &= \frac{-1}{\sqrt{2}}\left(y^7 + iy^8\right) \ , &\qquad Y^{2\dot{1}} &= \frac{-1}{\sqrt{2}}\left(y^7 - iy^8\right) \ .
\end{aligned}
\tag{B.13}
$$

The action of the SU(2) generators is a little complicated but we will be only interested in the bosons with Ω_3^+ charge $-\frac{1}{2}$, that is $Y_{1\dot{1}}$ and $Y_{2\dot{1}}$ and these satisfy

$$
\Omega_+^- \, Y_{2\dot{1}} = Y_{1\dot{1}} \quad , \quad \Omega_-^- \, Y_{1\dot{1}} = Y_{2\dot{1}} \ . \tag{B.14}
$$

In this notation the free part of the Lagrangian takes a similar form to that in (5.7)

$$
\begin{aligned}
\mathcal{L}_0 = \ & + \tfrac{1}{2}\dot{Y}^*_{a\dot{a}}\dot{Y}^{a\dot{a}} - \tfrac{1}{2}\acute{Y}^*_{a\dot{a}}\acute{Y}^{a\dot{a}} - \tfrac{1}{2}Y^*_{a\dot{a}}Y^{a\dot{a}} \\
& + i\Upsilon^*_{\alpha\dot{a}}\dot{\Upsilon}^{\alpha\dot{a}} + \tfrac{i}{2}(\Upsilon^*_{\alpha\dot{a}}\acute{\Upsilon}^{*\alpha\dot{a}} + \Upsilon_{\alpha\dot{a}}\acute{\Upsilon}^{\alpha\dot{a}}) + \Upsilon^*_{\alpha\dot{a}}\Upsilon^{\alpha\dot{a}} \ .
\end{aligned}
\tag{B.15}
$$

and so the equations of motion can be solved by a similar mode expansion:

$$
Y_{a\dot{a}}(\vec{x}) = \int \frac{dp}{2\pi}\frac{1}{\sqrt{2\varepsilon}}\left(a_{a\dot{a}}(p)\, e^{-i\vec{p}\cdot\vec{x}} + a^\dagger_{a\dot{a}}(p)\, e^{+i\vec{p}\cdot\vec{x}}\right) , \tag{B.16}
$$

$$
\Upsilon_{\alpha\dot{a}}(\vec{x}) = \int \frac{dp}{2\pi}\frac{1}{\sqrt{2\varepsilon}}\left(b_{\alpha\dot{a}}(p)\, u(p)\, e^{-i\vec{p}\cdot\vec{x}} + b^\dagger_{\alpha\dot{a}}(p)\, v(p)\, e^{+i\vec{p}\cdot\vec{x}}\right) , \tag{B.17}
$$

$$
\Upsilon^*_{\alpha\dot{a}}(\vec{x}) = \int \frac{dp}{2\pi}\frac{1}{\sqrt{2\varepsilon}}\left(b_{\alpha\dot{a}}(p)\, v(p)\, e^{-i\vec{p}\cdot\vec{x}} + b^\dagger_{\alpha\dot{a}}(p)\, u(p)\, e^{+i\vec{p}\cdot\vec{x}}\right) . \tag{B.18}
$$

The energy is still $\varepsilon = \sqrt{1+p^2}$ but the wave functions are slightly different than previously

$$
v(p) = \sqrt{2}\,\cosh\tfrac{\theta}{2} \quad , \quad u(p) = -\sqrt{2}\,\sinh\tfrac{\theta}{2} \ . \tag{B.19}
$$

The rapidity θ is still defined through $p = \sinh\theta$ and the scalar product in the exponentials is $\vec{p}\cdot\vec{x} = \varepsilon\tau + p\sigma$. The canonical commutation relations are given, as before, by

$$\begin{aligned}
&[a^{a\dot{a}}(p), a^{\dagger}_{b\dot{b}}(p')] = 2\pi\,\delta^a_b\delta^{\dot{a}}_{\dot{b}}\,\delta(p-p')\,, \qquad \{b^{a\dot{\alpha}}(p), b^{\dagger}_{b\dot{\beta}}(p')\} = 2\pi\,\delta^a_b\delta^{\dot{\alpha}}_{\dot{\beta}}\,\delta(p-p')\,,\\
&[a^{\alpha\dot{\alpha}}(p), a^{\dagger}_{\beta\dot{\beta}}(p')] = 2\pi\,\delta^\alpha_\beta\delta^{\dot{\alpha}}_{\dot{\beta}}\,\delta(p-p')\,, \quad \{b^{\alpha\dot{a}}(p), b^{\dagger}_{\beta\dot{b}}(p')\} = 2\pi\,\delta^\alpha_\beta\delta^{\dot{a}}_{\dot{b}}\,\delta(p-p')\,.
\end{aligned} \tag{B.20}$$

We focus on the fields $Y_{a\mathrm{i}}$ and $\Upsilon_{\alpha\mathrm{i}}$ which comprise a closed SU(2|2) subsector of the full theory and which makes comparison with Beisert's S-matrix most transparent. Parameterizing the T-matrix as

$$\mathrm{T}|Y_{a\mathrm{i}}Y'_{b\mathrm{i}}\rangle = \mathrm{A}(p,p')|Y_{a\mathrm{i}}Y'_{b\mathrm{i}}\rangle + \mathrm{B}(p,p')|Y_{b\mathrm{i}}Y'_{a\mathrm{i}}\rangle + \mathrm{C}(p,p')\epsilon_{ab}\epsilon^{\alpha\beta}|\Upsilon_{\alpha\mathrm{i}}\Upsilon'_{\beta\mathrm{i}}\rangle \tag{B.21}$$

$$\mathrm{T}|Y_{a\mathrm{i}}\Upsilon'_{\beta\mathrm{i}}\rangle = \mathrm{G}(p,p')|Y_{a\mathrm{i}}\Upsilon'_{\beta\mathrm{i}}\rangle + \mathrm{H}(p,p')|\Upsilon_{\beta\mathrm{i}}Y'_{a\mathrm{i}}\rangle \tag{B.22}$$

$$\mathrm{T}|\Upsilon_{\alpha\mathrm{i}}Y'_{b\mathrm{i}}\rangle = \mathrm{K}(p,p')|Y_{b\mathrm{i}}\Upsilon'_{\alpha\mathrm{i}}\rangle + \mathrm{L}(p,p')|\Upsilon_{\alpha\mathrm{i}}Y'_{b\mathrm{i}}\rangle \tag{B.23}$$

$$\mathrm{T}|\Upsilon_{\alpha\mathrm{i}}\Upsilon'_{\beta\mathrm{i}}\rangle = \mathrm{D}(p,p')|\Upsilon_{\alpha\mathrm{i}}\Upsilon'_{\beta\mathrm{i}}\rangle + \mathrm{E}(p,p')|\Upsilon_{\beta\mathrm{i}}\Upsilon'_{\alpha\mathrm{i}}\rangle + \mathrm{F}(p,p')\epsilon_{\alpha\beta}\epsilon^{ab}|Y_{a\mathrm{i}}Y'_{b\mathrm{i}}\rangle\,, \tag{B.24}$$

we find

$$\mathrm{A}(p,p') = \frac{1}{2}\left[\varepsilon' p - \varepsilon p' + \frac{p'^2+p^2}{\varepsilon' p - \varepsilon p'}\right], \tag{B.25}$$

$$\mathrm{B}(p,p') = \mathrm{E}(p,p') = \frac{pp'}{\varepsilon' p - \varepsilon p'}\,, \tag{B.26}$$

$$\mathrm{C}(p,p') = \mathrm{F}(p,p') = -\frac{1}{2}\,\frac{\sqrt{(\varepsilon+1)(\varepsilon'+1)}\,(\varepsilon' p - \varepsilon p' + p' - p)}{\varepsilon' p - \varepsilon p'}\,, \tag{B.27}$$

$$\mathrm{D}(p,p') = \frac{1}{2}\left[\varepsilon' p - \varepsilon p' - \frac{2pp'}{\varepsilon' p - \varepsilon p'}\right], \tag{B.28}$$

$$\mathrm{G}(p,p') = \mathrm{L}(p',p) = \frac{1}{2}\left[\varepsilon' p - \varepsilon p' + \frac{(p+p')p'}{\varepsilon' p - \varepsilon p'}\right], \tag{B.29}$$

$$\mathrm{H}(p,p') = \mathrm{K}(p,p') = \frac{1}{2}\,\frac{pp'}{\varepsilon' p - \varepsilon p'}\,\frac{(\varepsilon+1)(\varepsilon'+1) - pp'}{\sqrt{(\varepsilon'+1)(\varepsilon+1)}}\,. \tag{B.30}$$

We note that in this case, as we are explicitly restricting our fields to lie in a single $\mathfrak{su}(2|2)$ rather than calculating the factorized T-matrix, there is no additional $\frac{1}{2}(\mathrm{A}-\mathrm{B})\ (\mathbb{1}\otimes\mathbb{1})$ which must be included to get the expressions used in section 2. There is an ambiguity in the sign of F which is due to the choice of the fermion ordering; here we have used the convention

$$|\Upsilon_{\alpha\mathrm{i}}\Upsilon'_{\beta\mathrm{i}}\rangle = b^{\dagger}_{\alpha\mathrm{i}}b'^{\dagger}_{\beta\mathrm{i}}|J_+\rangle \tag{B.31}$$

and so it follows from the hermiticity of the Hamiltonian that F = C. These expressions are in good agreement with the gauge theory, there are of course the terms linear in momenta which are presumably related to the difference between the definition of the string length and that of the gauge theory spin chain.

Now, we construct the supersymmetry generators in this $\mathfrak{su}(2|2)$ sector. The analogous calculation for the uniform gauge was explicitly carried out in [30] and we will here repeat

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their calculation for the constant-J gauge, at least to lowest order. We will start with the Noether currents corresponding to left multiplication in the gauge unfixed theory and give expressions in terms of all ten bosonic coordinates, x^μ, and the sixteen component complex spinor θ. We can then gauge fix these currents to find their action on the physical fields which are scattered by the S-matrix. The Noether currents are given by $j = p + *q + *\bar{q}$ where

$$\begin{aligned} j &= g(x,\theta)\ J\ g(x,\theta)^{-1} \\ &= g(x,\theta)\left(L^A P_A + *L^\alpha Q_\alpha + *\bar{L}^\alpha \bar{Q}_\alpha\right) g(x,\theta)^{-1} \end{aligned} \tag{B.32}$$

and $g(x,\theta) = \exp(\frac{1}{2}(x^+P^- + x^-P^+))\exp(x^I P^I)\exp(\theta\bar{Q} + \bar{\theta}Q)$. For compactness it is useful to define $\theta^\alpha F_\alpha = \theta^\alpha \bar{Q}_\alpha + \bar{\theta}^\alpha Q_\alpha$ and introduce the quantities

$$\begin{aligned} \pi_A(\theta) &= \mathrm{e}^{\theta F} P_A \mathrm{e}^{-\theta F} \\ &= \pi_A^B P_A + \pi_A^\alpha F_\alpha \text{ and} \\ \pi_\alpha(\theta) &= \mathrm{e}^{\theta F} F_\alpha \mathrm{e}^{-\theta F} \\ &= \pi_\alpha^B P_A + \pi_\alpha^\beta F_\beta \end{aligned} \tag{B.33}$$

so that

$$j = g(x)\left(\left(L^A \pi_A^B + *L^\alpha \pi_\alpha^B\right) P_B + \left(L^A \pi_A^\beta + *L^\alpha \pi_\alpha^\beta\right) F_\beta\right) g(x)^{-1}. \tag{B.34}$$

Now, using the usual trick of scaling the fermions, $\theta \to t\theta$, taking the derivative and integrating using the boundary conditions

$$\begin{aligned} &\pi_B^A(t=0) = \delta_B^A \qquad \pi_A^\alpha(t=0) = 0 \\ &\pi_\alpha^A(t=0) = 0 \qquad \pi_\beta^\alpha(t=0) = \delta_\beta^\alpha \end{aligned} \tag{B.35}$$

we can find the closed expressions

$$\pi_B^A = \cos\left(\sqrt{\alpha\beta}\right), \qquad \pi_A^\alpha = \frac{\sin\sqrt{\beta\alpha}}{\sqrt{\beta\alpha}}\beta \tag{B.36}$$

$$\pi_\alpha^A = -\frac{\sin\sqrt{\alpha\beta}}{\sqrt{\alpha\beta}}\alpha \qquad \pi_\alpha^\beta = \cos\sqrt{\beta\alpha} \tag{B.37}$$

with the short hand

$$\beta_A^\alpha = f_{\gamma A}^\alpha \theta^\gamma\ , \qquad \alpha_\beta^A = \theta^\gamma f_{\gamma\beta}^A \tag{B.38}$$

and the $f_{(A,\alpha)(B,\beta)}^{(C,\gamma)}$ are the $\mathfrak{psu}(2,2|4)$ structure constants. We are particularly interested in the current corresponding to the conserved charges $Q^- = \frac{1}{2}\bar{\gamma}^+\gamma^- Q$ so we consider the truncation

$$\begin{aligned} \mathcal{Q}^- &= j|_{Q^-} \\ &= \frac{1}{2}\exp\left(\frac{-ix^- \Pi}{2}\right)\exp\left(\frac{ix^I}{2}\bar{\gamma}^0 \Pi \bar{\gamma}^I\right)\left(\pi_A^\alpha L^A + \pi_\beta^\alpha * L\right). \end{aligned} \tag{B.39}$$

We have used the $\mathfrak{psu}(2,2|4)$ algebra, in particular the relation

$$[Q, P^\mu] = \frac{i}{2} Q\bar{\gamma}^0 \Pi \bar{\gamma}^\mu, \qquad \Pi = \gamma^1\bar{\gamma}^2\gamma^3\bar{\gamma}^4 \ . \tag{B.40}$$

which implies for our choice of coset representative that

$$g(x)Q^- g(x) = \tfrac{1}{2} Q^- \exp\left(\frac{-ix^-\Pi}{2}\right) \exp\left(\frac{ix^I}{2}\bar{\gamma}^0\Pi\bar{\gamma}^I\right) \ . \tag{B.41}$$

The most important result is the occurrence of the $e^{i\frac{x^-\Pi}{2}}$ factor in the definition of the Noether current. As discussed in section 3 it is this factor which is responsible for the non-trivial coproduct and hence the non-trivial realization of integrability. It is worth noting that this factor does not occur in the pp-wave background as there $[P^+, Q^-] = 0$. In order to get manageable expressions and to check that we have sensible results we expand the time component of the current in powers of the physical fields and keep only the lowest, quadratic, part

$$\begin{aligned} \pi_\alpha^\beta = \delta_\alpha^\beta \ , \qquad \bar{\pi}_A^\alpha = -\frac{i}{2}\bar{\gamma}^0\Pi\bar{\gamma}_A\bar{\theta} \\ L^+ = 2dx^+ \ , \ L^I = dx^I, \ L = d\theta + idx^+\Pi\theta \end{aligned} \tag{B.42}$$

$$\begin{aligned} \bar{Q}_0^- &= e^{\frac{-ix^-\Pi}{2}} e^{\frac{ix^I\bar{\gamma}^0\Pi\bar{\gamma}^I}{2}} \left(\bar{\pi}_A^\alpha L_0^A + \pi_\beta^\alpha L_1\right) \\ &= -\frac{i}{2} e^{\frac{-ix^-\Pi}{2}} \Pi(p^I\bar{\gamma}^I\bar{\theta} - ix^I\bar{\gamma}^I\Pi\bar{\theta} + x^I\bar{\gamma}^I\acute{\theta}) \end{aligned} \tag{B.43}$$

which we can compare with the results of Metsaev for the total charge [24] (up to an overall normalization)

$$\bar{Q}_{p.p.}^- = \int d\sigma \ \left(p^I\bar{\gamma}^I\bar{\theta} - ix^I\bar{\gamma}^I\Pi\bar{\theta} - \acute{x}^I\bar{\gamma}^I\theta\right) \tag{B.44}$$

and which agrees with our result if we drop the $e^{i\frac{x^-\Pi}{2}}$ and integrate the last term by parts; there is of course a similar expression for the conjugate supercharge. It is interesting to further note that even in the plane-wave geometry there is a central extension of the $\mathfrak{psu}(2,2|4)$ algebra as can be easily seen if we calculate the Poisson bracket of two holomorphic or anti-holomorphic supercharges

$$\begin{aligned} \mathrm{tr}\{Q^-, Q^-\} &\propto \int d\sigma \ \left(p^I\acute{x}^I + i\bar{\theta}\acute{\theta}\right) \\ &= -\int d\sigma \ \acute{x}^- \end{aligned} \tag{B.45}$$

using the constraint equation in the last line. However in the plane wave limit there is no non-trivial coproduct as there is no non-local e^{ix^-} term. We can further restrict our charges so that they lie in a single $\mathfrak{su}(2|2)$ by imposing $\Pi Q^- = -Q^-$ so that they now only depend on the fermionic fields $\Upsilon_{\alpha \mathrm{i}}$. In the full geometry the charges are $Q^- =$

$\int d\sigma e^{\pm\frac{i}{2}x^-}\,\Omega\,(Y, Y^*, \Upsilon, \Upsilon^*)$ where Ω is a local function of the physical fields and including the effect of the exponential factor gives rise to the non-trivial phase factor, cf. section 3. We note that even at higher orders in fields there are no additional non-local terms depending on x^- and so the effects of the non-trivial coproduct are entirely captured by including the e^{ix^-} terms.

C. Rewriting the uniform light-cone gauge action

For the superstring computation in uniform light-cone gauge, we make use of the result of [29]. The authors of that paper wrote the Green-Schwarz superstring in a first order formalism and fixed the uniform light-cone gauge and the kappa-symmetry. In order to quantize the theory, they considered the near-plane wave limit. The Lagrangian was expanded in the transverse fields and the fermions were shifted $\chi \mapsto \chi + \Phi(p, x, \chi)$ to obtain a canonical kinetic term. Furthermore the fields were rescaled approriately and a canonical transformation was applied to the bosonic sector to remove all non-derivative quartic terms. The results we are interested in are given in (5.4) with rescaling (5.6), in (5.13), and in (5.16) of [29]. In the notation of [29], the Green-Schwarz superstring in the uniform light-cone gauge up to quartic[19] order in the fields reads

$$\mathcal{S} = \int_{-\infty}^{\infty} d\tau \int_{-\pi}^{\pi} \frac{d\sigma}{2\pi}\,\mathcal{L} \quad , \quad \mathcal{L} = \mathcal{L}_{\rm kin} - \mathcal{H} \quad , \quad \mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 \tag{C.1}$$

with[20]

$$\begin{aligned}
\mathcal{L}_{\rm kin} &= p_M \dot{x}_M - \frac{i}{2}\,\mathrm{str}\big(\Sigma_+\chi\dot{\chi}\big) \\
\mathcal{H}_2 &= \frac{1}{2}p_M^2 + \frac{\tilde{\lambda}}{2}\acute{x}_M^2 + \frac{1}{2}x_M^2 + \frac{\kappa\sqrt{\tilde{\lambda}}}{2}\,\mathrm{str}\big(\Sigma_+\chi\widetilde{K}_8\acute{\chi}^t K_8\big) + \frac{1}{2}\,\mathrm{str}\big(\chi^2\big) \\
\mathcal{H}_4 &= \frac{1}{2P_+}\Bigg[2\tilde{\lambda}\left(\acute{y}^2 z^2 - \acute{z}^2 y^2 + \acute{z}^2 z^2 - \acute{y}^2 y^2\right) \\
&\quad - \tilde{\lambda}\,\mathrm{str}\left(\frac{1}{2}\chi\acute{\chi}\chi\acute{\chi} + \frac{1}{2}\chi^2\acute{\chi}^2 + \frac{1}{4}[\chi,\acute{\chi}]K_8[\chi,\acute{\chi}]^t K_8 + \chi\widetilde{K}_8\acute{\chi}^t K_8\chi\widetilde{K}_8\acute{\chi}^t K_8\right) \\
&\quad + \tilde{\lambda}\,\mathrm{str}\left((z^2 - y^2)\acute{\chi}\acute{\chi} + \frac{1}{2}\acute{x}_M x_N[\Sigma_M, \Sigma_N][\chi,\acute{\chi}] - 2x_M x_N\Sigma_M\acute{\chi}\Sigma_N\acute{\chi}\right) \\
&\quad + \frac{i\kappa\sqrt{\tilde{\lambda}}}{4}(x_M p_N)'\,\mathrm{str}\big([\Sigma_M, \Sigma_N][\widetilde{K}_8\chi^t K_8, \chi]\big)\Bigg]\,.
\end{aligned} \tag{C.2}$$

Here

$$\tilde{\lambda} = \frac{4\lambda}{P_+^2} \tag{C.3}$$

[19] A discussion of the Dirac brackets in light-cone gauge to all orders in fields has appeared in [54].

[20] In formula (5.16) of [29] there is actually a factor of $\frac{1}{2}$ missing in front of the second term in the second line.

is the effective coupling constant which is kept finite in the plane-wave limit $P_+ \to \infty$. The parameter $P_+ := J + E$ itself defines the light-cone gauge, and corresponds to $P_+ = 2J_+$ in our conventions (4.2).

All gauge symmetries are fixed in (C.2) and we are left with 16 real bosonic and 16 real fermionic degrees of freedom given by the following fields. The bosonic coordinates and their canonical conjugate momenta are denoted by

$$x_M, p_M \quad , \quad M = 1, \ldots, 8 \,. \tag{C.4}$$

These are the coordinates transverse to the light-cone. They are divided into coordinates z_a with $a = 1, \ldots, 4$ on AdS and coordinates y_s with $s = 1, \ldots, 4$ on S^5. The (complex) fermionic variables are contained in the matrix

$$\chi = \begin{pmatrix} 0 & \Theta \\ -\Theta^\dagger \Sigma & 0 \end{pmatrix} \quad , \quad \Theta = \begin{pmatrix} 0 & 0 & \theta_{13} & \theta_{14} \\ 0 & 0 & \theta_{23} & \theta_{24} \\ \theta_{31} & \theta_{32} & 0 & 0 \\ \theta_{41} & \theta_{42} & 0 & 0 \end{pmatrix} . \tag{C.5}$$

The various constant matrices Σ and K used in these formulas are defined as follows:

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad , \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
$$\gamma_4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad , \quad \gamma_5 = \Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{C.6}$$

$$\Sigma_M = \left(\begin{pmatrix} \gamma_a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & i\gamma_s \end{pmatrix} \right) \tag{C.7}$$

$$\Sigma_+ = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix} \quad , \quad \Sigma_- = \begin{pmatrix} -\Sigma & 0 \\ 0 & \Sigma \end{pmatrix} \quad , \quad \Sigma_8 = -\Sigma_+ \Sigma_- = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix} \tag{C.8}$$

$$K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad , \quad K_8 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \quad , \quad \widetilde{K}_8 = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix} . \tag{C.9}$$

We will now change back to a second order formalism. Using $\dot{x}_M = \partial \mathcal{H} / \partial p_M$ we find the momentum to cubic order in the fields

$$p_M = \dot{x}_M + \frac{i\kappa\sqrt{\tilde{\lambda}}}{8J_+} x_N \partial_\sigma \operatorname{str}\big([\Sigma_N, \Sigma_M][\widetilde{K}_8 \chi^t K_8, \chi]\big) \,. \tag{C.10}$$

Plugging this into the Lagrangian yields

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{int}} \tag{C.11}$$

with

$$\mathcal{L}_0 = \frac{1}{2}\dot{x}_M^2 - \frac{\lambda}{2J_+^2}\acute{x}_M^2 - \frac{1}{2}x_M^2 - \frac{i}{2}\,\mathrm{str}\big(\Sigma_+\chi\dot{\chi}\big) - \frac{\kappa\sqrt{\lambda}}{2J_+}\,\mathrm{str}\big(\Sigma_+\chi\acute{\chi}^\natural\big) - \frac{1}{2}\,\mathrm{str}\big(\chi^2\big)$$

$$\begin{aligned}
\mathcal{L}_{\mathrm{int}} = &-\frac{\lambda}{2J_+^3}\left(\acute{y}^2z^2 - \acute{z}^2y^2 + \acute{z}^2z^2 - \acute{y}^2y^2\right)\\
&+\frac{\lambda}{4J_+^3}\,\mathrm{str}\left(\frac{1}{2}\chi\acute{\chi}\chi\acute{\chi} + \frac{1}{2}\chi^2\acute{\chi}^2 + \frac{1}{4}[\chi,\acute{\chi}][\chi^\natural,\acute{\chi}^\natural] + \chi\acute{\chi}^\natural\chi\acute{\chi}^\natural\right)\\
&-\frac{\lambda}{4J_+^3}\,\mathrm{str}\left((z^2-y^2)\acute{\chi}\acute{\chi} + \frac{1}{2}\acute{x}_M x_N[\Sigma_M,\Sigma_N][\chi,\acute{\chi}] - 2x_Mx_N\Sigma_M\acute{\chi}\Sigma_N\acute{\chi}\right)\\
&+\frac{i\kappa\sqrt{\lambda}}{16J_+^2}x_M\dot{x}_N\partial_\sigma\,\mathrm{str}\big([\Sigma_M,\Sigma_N][\chi^\natural,\chi]\big)
\end{aligned} \tag{C.12}$$

where we used $\tilde{\lambda} = \lambda/J_+^2$ and introduced the conjugation $()^\natural$. For bosonic (X) and fermionic (χ) supermatrices

$$X = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \quad , \quad \chi = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \tag{C.13}$$

this is defined as

$$X^\natural := K_8X^tK_8 = \begin{pmatrix} KA^tK & 0 \\ 0 & KD^tK \end{pmatrix} \quad , \quad \chi^\natural := \widetilde{K}_8\chi^tK_8 = \begin{pmatrix} 0 & KC^tK \\ -KB^tK & 0 \end{pmatrix} . \tag{C.14}$$

If the bosonic matrix is a product of fermionic ones, we can use $(\chi_1\chi_2)^\natural = -\chi_2^\natural\chi_1^\natural$.

To clean up the notation, we finally put the bosonic degrees of freedom into a supermatrix

$$X := x_M\Sigma_M\ , \tag{C.15}$$

rescale $X \to \sqrt{2J_+}\,X$, $\chi \to \sqrt{J_+}\,\chi$, $\sigma \to \sqrt{\lambda}/J_+\,\sigma$ and fix $\kappa = 1$. Then the action takes the form (5.3) given in the main text.

D. $\mathrm{SU}(2)^4$ T-matrix in uniform light-cone gauge

Here we list our results for the full T-matrix in uniform light-cone gauge. There are some identities, which are useful in this context:

$$\begin{aligned}
&\varepsilon' p - \varepsilon p' = \sinh(\theta - \theta') \\
&(p - p')\cosh\tfrac{\theta-\theta'}{2} = (\varepsilon + \varepsilon')\sinh\tfrac{\theta-\theta'}{2} \\
&\sinh\tfrac{\theta}{2} = \tfrac{1}{2}\sqrt{\varepsilon + p} - \tfrac{1}{2}\sqrt{\varepsilon - p} \\
&\cosh\tfrac{\theta}{2} = \tfrac{1}{2}\sqrt{\varepsilon + p} + \tfrac{1}{2}\sqrt{\varepsilon - p} \\
&\sinh\tfrac{\theta-\theta'}{2} = \tfrac{1}{2}\sqrt{(\varepsilon+p)(\varepsilon'-p')} - \tfrac{1}{2}\sqrt{(\varepsilon-p)(\varepsilon'+p')} \\
&\cosh\tfrac{\theta-\theta'}{2} = \tfrac{1}{2}\sqrt{(\varepsilon+p)(\varepsilon'-p')} + \tfrac{1}{2}\sqrt{(\varepsilon-p)(\varepsilon'+p')}
\end{aligned} \tag{D.1}$$

Boson-Boson

$$\begin{aligned}
\mathbb{T}|Y_{a\dot{a}}Y'_{b\dot{b}}\rangle =\ & +\tfrac{1}{2}\tfrac{(p-p')^2}{\varepsilon'p-\varepsilon p'}|Y_{a\dot{a}}Y'_{b\dot{b}}\rangle + \tfrac{pp'}{\varepsilon'p-\varepsilon p'}\Big(|Y_{a\dot{b}}Y'_{b\dot{a}}\rangle + |Y_{b\dot{a}}Y'_{a\dot{b}}\rangle\Big) \\
& - \tfrac{pp'}{\varepsilon'p-\varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\Big(\epsilon_{\dot{a}\dot{b}}\epsilon^{\dot{\alpha}\dot{\beta}}|\Psi_{a\dot{\alpha}}\Psi'_{b\dot{\beta}}\rangle + \epsilon_{ab}\epsilon^{\alpha\beta}|\Upsilon_{\alpha\dot{a}}\Upsilon'_{\beta\dot{b}}\rangle\Big) \\
\mathbb{T}|Z_{\alpha\dot{\alpha}}Z'_{\beta\dot{\beta}}\rangle =\ & -\tfrac{1}{2}\tfrac{(p-p')^2}{\varepsilon'p-\varepsilon p'}|Z_{\alpha\dot{\alpha}}Z'_{\beta\dot{\beta}}\rangle - \tfrac{pp'}{\varepsilon'p-\varepsilon p'}\Big(|Z_{\alpha\dot{\beta}}Z'_{\beta\dot{\alpha}}\rangle + |Z_{\beta\dot{\alpha}}Z'_{\alpha\dot{\beta}}\rangle\Big) \\
& + \tfrac{pp'}{\varepsilon'p-\varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\Big(\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{a}\dot{b}}|\Upsilon_{\alpha\dot{a}}\Upsilon'_{\beta\dot{b}}\rangle + \epsilon_{\alpha\beta}\epsilon^{ab}|\Psi_{a\dot{\alpha}}\Psi'_{b\dot{\beta}}\rangle\Big) \\
\mathbb{T}|Y_{a\dot{a}}Z'_{\alpha\dot{\alpha}}\rangle =\ & -\tfrac{1}{2}\tfrac{p^2-p'^2}{\varepsilon'p-\varepsilon p'}|Y_{a\dot{a}}Z'_{\alpha\dot{\alpha}}\rangle + \tfrac{pp'}{\varepsilon'p-\varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}\Big(|\Upsilon_{\alpha\dot{a}}\Psi'_{a\dot{\alpha}}\rangle - |\Psi_{a\dot{\alpha}}\Upsilon'_{\alpha\dot{a}}\rangle\Big) \\
\mathbb{T}|Z_{\alpha\dot{\alpha}}Y'_{a\dot{a}}\rangle =\ & +\tfrac{1}{2}\tfrac{p^2-p'^2}{\varepsilon'p-\varepsilon p'}|Z_{\alpha\dot{\alpha}}Y'_{a\dot{a}}\rangle - \tfrac{pp'}{\varepsilon'p-\varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}\Big(|\Psi_{a\dot{\alpha}}\Upsilon'_{\alpha\dot{a}}\rangle - |\Upsilon_{\alpha\dot{a}}\Psi'_{a\dot{\alpha}}\rangle\Big)
\end{aligned} \tag{D.2}$$

Fermion-Fermion

$$\begin{aligned}
\mathbb{T}|\Psi_{a\dot{\alpha}}\Psi'_{b\dot{\beta}}\rangle =\ & +\tfrac{pp'}{\varepsilon'p-\varepsilon p'}\Big(|\Psi_{b\dot{\alpha}}\Psi'_{a\dot{\beta}}\rangle - |\Psi_{a\dot{\beta}}\Psi'_{b\dot{\alpha}}\rangle\Big) \\
& - \tfrac{pp'}{\varepsilon'p-\varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\Big(\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{a}\dot{b}}|Y_{a\dot{a}}Y'_{b\dot{b}}\rangle - \epsilon_{ab}\epsilon^{\alpha\beta}|Z_{\alpha\dot{\alpha}}Z'_{\beta\dot{\beta}}\rangle\Big) \\
\mathbb{T}|\Upsilon_{\alpha\dot{a}}\Upsilon'_{\beta\dot{b}}\rangle =\ & -\tfrac{pp'}{\varepsilon'p-\varepsilon p'}\Big(|\Upsilon_{\beta\dot{a}}\Upsilon'_{\alpha\dot{b}}\rangle - |\Upsilon_{\alpha\dot{b}}\Upsilon'_{\beta\dot{a}}\rangle\Big) \\
& + \tfrac{pp'}{\varepsilon'p-\varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\Big(\epsilon_{\dot{a}\dot{b}}\epsilon^{\dot{\alpha}\dot{\beta}}|Z_{\alpha\dot{\alpha}}Z'_{\beta\dot{\beta}}\rangle - \epsilon_{\alpha\beta}\epsilon^{ab}|Y_{a\dot{a}}Y'_{b\dot{b}}\rangle\Big) \\
\mathbb{T}|\Psi_{a\dot{\alpha}}\Upsilon'_{\beta\dot{b}}\rangle =\ & -\tfrac{pp'}{\varepsilon'p-\varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}\Big(|Y_{a\dot{b}}Z'_{\beta\dot{\alpha}}\rangle + |Z_{\beta\dot{\alpha}}Y'_{a\dot{b}}\rangle\Big) \\
\mathbb{T}|\Upsilon_{\alpha\dot{a}}\Psi'_{b\dot{\beta}}\rangle =\ & +\tfrac{pp'}{\varepsilon'p-\varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}\Big(|Z_{\alpha\dot{\beta}}Y'_{b\dot{a}}\rangle + |Y_{b\dot{a}}Z'_{\alpha\dot{\beta}}\rangle\Big)
\end{aligned} \tag{D.3}$$

Boson-Fermion

$$
\begin{aligned}
\mathbb{T}|Y_{a\dot a}\mathit{\Psi}'_{b\dot\beta}\rangle =\ & +\tfrac{1}{2}\tfrac{(p'-p)p'}{\varepsilon' p-\varepsilon p'}|Y_{a\dot a}\mathit{\Psi}'_{b\dot\beta}\rangle+\tfrac{pp'}{\varepsilon' p-\varepsilon p'}|Y_{b\dot a}\mathit{\Psi}'_{a\dot\beta}\rangle\\
& +\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}|\mathit{\Psi}_{a\dot\beta}Y'_{b\dot a}\rangle-\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\epsilon_{ab}\epsilon^{\alpha\beta}|\mathit{\Upsilon}_{\alpha\dot a}Z'_{\beta\dot\beta}\rangle\\
\mathbb{T}|Y_{a\dot a}\mathit{\Upsilon}'_{\beta\dot b}\rangle =\ & +\tfrac{1}{2}\tfrac{(p'-p)p'}{\varepsilon' p-\varepsilon p'}|Y_{a\dot a}\mathit{\Upsilon}'_{\beta\dot b}\rangle+\tfrac{pp'}{\varepsilon' p-\varepsilon p'}|Y_{a\dot b}\mathit{\Upsilon}'_{\beta\dot a}\rangle\\
& +\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}|\mathit{\Upsilon}_{\beta\dot a}Y'_{a\dot b}\rangle+\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\epsilon_{\dot a\dot b}\epsilon^{\dot\alpha\dot\beta}|\mathit{\Psi}_{a\dot\alpha}Z'_{\beta\dot\beta}\rangle\\
\mathbb{T}|\mathit{\Psi}_{a\dot\alpha}Y'_{b\dot b}\rangle =\ & +\tfrac{1}{2}\tfrac{(p-p')p}{\varepsilon' p-\varepsilon p'}|\mathit{\Psi}_{a\dot\alpha}Y'_{b\dot b}\rangle+\tfrac{pp'}{\varepsilon' p-\varepsilon p'}|\mathit{\Psi}_{b\dot\alpha}Y'_{a\dot b}\rangle\\
& +\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}|Y_{a\dot b}\mathit{\Psi}'_{b\dot\alpha}\rangle+\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\epsilon_{ab}\epsilon^{\alpha\beta}|Z_{\alpha\dot\alpha}\mathit{\Upsilon}'_{\beta\dot b}\rangle\\
\mathbb{T}|\mathit{\Upsilon}_{\alpha\dot a}Y'_{b\dot b}\rangle =\ & +\tfrac{1}{2}\tfrac{(p-p')p}{\varepsilon' p-\varepsilon p'}|\mathit{\Upsilon}_{\alpha\dot a}Y'_{b\dot b}\rangle+\tfrac{pp'}{\varepsilon' p-\varepsilon p'}|\mathit{\Upsilon}_{\alpha\dot b}Y'_{b\dot a}\rangle\\
& +\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}|Y_{b\dot a}\mathit{\Upsilon}'_{\alpha\dot b}\rangle-\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\epsilon_{\dot a\dot b}\epsilon^{\dot\alpha\dot\beta}|Z_{\alpha\dot\alpha}\mathit{\Psi}'_{b\dot\beta}\rangle
\end{aligned}
\tag{D.4}
$$

$$
\begin{aligned}
\mathbb{T}|Z_{\alpha\dot\alpha}\mathit{\Psi}'_{b\dot\beta}\rangle =\ & -\tfrac{1}{2}\tfrac{(p'-p)p'}{\varepsilon' p-\varepsilon p'}|Z_{\alpha\dot\alpha}\mathit{\Psi}'_{b\dot\beta}\rangle-\tfrac{pp'}{\varepsilon' p-\varepsilon p'}|Z_{\alpha\dot\beta}\mathit{\Psi}'_{b\dot\alpha}\rangle\\
& -\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}|\mathit{\Psi}_{b\dot\alpha}Z'_{\alpha\dot\beta}\rangle-\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot a\dot b}|\mathit{\Upsilon}_{\alpha\dot a}Y'_{b\dot b}\rangle\\
\mathbb{T}|Z_{\alpha\dot\alpha}\mathit{\Upsilon}'_{\beta\dot b}\rangle =\ & -\tfrac{1}{2}\tfrac{(p'-p)p'}{\varepsilon' p-\varepsilon p'}|Z_{\alpha\dot\alpha}\mathit{\Upsilon}'_{\beta\dot b}\rangle-\tfrac{pp'}{\varepsilon' p-\varepsilon p'}|Z_{\beta\dot\alpha}\mathit{\Upsilon}'_{\alpha\dot b}\rangle\\
& -\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}|\mathit{\Upsilon}_{\alpha\dot b}Z'_{\beta\dot\alpha}\rangle+\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\epsilon_{\alpha\beta}\epsilon^{ab}|\mathit{\Psi}_{a\dot\alpha}Y'_{b\dot b}\rangle\\
\mathbb{T}|\mathit{\Psi}_{a\dot\alpha}Z'_{\beta\dot\beta}\rangle =\ & -\tfrac{1}{2}\tfrac{(p-p')p}{\varepsilon' p-\varepsilon p'}|\mathit{\Psi}_{a\dot\alpha}Z'_{\beta\dot\beta}\rangle-\tfrac{pp'}{\varepsilon' p-\varepsilon p'}|\mathit{\Psi}_{a\dot\beta}Z'_{\beta\dot\alpha}\rangle\\
& -\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}|Z_{\beta\dot\alpha}\mathit{\Psi}'_{a\dot\beta}\rangle+\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot a\dot b}|Y_{a\dot a}\mathit{\Upsilon}'_{\beta\dot b}\rangle\\
\mathbb{T}|\mathit{\Upsilon}_{\alpha\dot a}Z'_{\beta\dot\beta}\rangle =\ & -\tfrac{1}{2}\tfrac{(p-p')p}{\varepsilon' p-\varepsilon p'}|\mathit{\Upsilon}_{\alpha\dot a}Z'_{\beta\dot\beta}\rangle-\tfrac{pp'}{\varepsilon' p-\varepsilon p'}|\mathit{\Upsilon}_{\beta\dot a}Z'_{\alpha\dot\beta}\rangle\\
& -\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\cosh\tfrac{\theta-\theta'}{2}|Z_{\alpha\dot\beta}\mathit{\Upsilon}'_{\beta\dot a}\rangle-\tfrac{pp'}{\varepsilon' p-\varepsilon p'}\sinh\tfrac{\theta-\theta'}{2}\epsilon_{\alpha\beta}\epsilon^{ab}|Y_{a\dot a}\mathit{\Psi}'_{b\dot\beta}\rangle
\end{aligned}
\tag{D.5}
$$

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